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ONE-SIDED INEQUALITIES FOR THE SUCCESSIVE DERIVATIVES OF A FUNCTION

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We fix an integer $n \geqslant 2$ and consider the class F of bounded continuous functions on the positive x-axis satisfying

(i) $-1 \leqslant f(x) \leqslant 1$ for $x \in R^+$,

(ii) $f^{(n-1)}(x)$ is absolutely continuous on R^+,

(iii) $f^{(n)}(x) \leqslant 1$ a.e. on R^+.

Under these conditions on f, our goal is to establish best possible inequalities for the intermediate derivatives $f^{(j)}(x)$. We thus extend the work begun by Landau, further developed by Schoenberg and Cavaretta, and by Hörmander [4], [5], [2].

To settle our question for the class F, we need, as extremal functions, the monosplines of R. S. Johnson. A monospline of degree n with k knots is a function of the form

$$M(x) = \frac{x^n}{n!} + \sum_{i=0}^{n-1} a_i x^i + \sum_{i=1}^{k} c_i (x - \xi_i)_+^{n-1}$$

where the a_i, c_i, and ξ_i are freely chosen real parameters. We note that $M^{(n-1)}(x)$ consists of $k + 1$ straight line segments, each of slope 1. Considering such monosplines restricted to $[-1, 1]$, Johnson [3] proves the following

THEOREM. *There exists a uniquely determined monospline $M_{n,k}(x)$ having precisely $n + 2k + 1$ points of equioscillation on $[-1, 1]$. In addition, $M_{n,k}$ has least* sup norm *on $[-1, 1]$; i.e., $\|M_{n,k}\|_\infty \leqslant \|M\|_\infty$ with equality only if $M = M_{n,k}$.*

The relevance of these functions to our class F becomes apparent after we make a preliminary change of scale and origin. We consider $f(x) = aM_{n,k}(bx)$ ($a > 0$, $b > 0$) and determine $a = a_{n,k}$ and $b = b_{n,k}$ so that $\|f\|_\infty = 1$ on $[-b^{-1}, b^{-1}]$ and $f^{(n)}(x) = 1$ except at its knots. Then define $B_{n,k}(x) = aM_{n,k}(bx - 1)$ on the interval $[0, 2b^{-1}]$. In this fashion we obtain the monospline $B_{n,k}$ which on $[0, 2b^{-1}]$ is of norm 1 and has precisely $n + 2k + 1$ points of equioscillation there. By elementary zero counting arguments, one readily verifies that $\operatorname{sign} B_{n,k}^{(j)}(0) = (-1)^{n+j}$.

With these preliminaries, we can now state the main theorem and its corollary.

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THEOREM. *Let* $1 \leqslant j \leqslant n-1$. *For any* $f \in F$ *and any* $x \geqslant 0$ *we have* (1) *if* $n+j$ *is odd, then* $f^{(j)}(x) \geqslant B_{n,k}^{(j)}(0)$ *and no finite upper bound for* $f^{(j)}(x)$ *is possible*; (2) *if* $n+j$ *is even, then* $f^{(j)}(x) \leqslant B_{n,k}^{(j)}(0)$ *and no finite lower bound for* $f^{(j)}(x)$ *is possible.*

COROLLARY. *The limit* $\beta_n^j = \lim_{k\to\infty} B_{n,k}^{(j)}(0)$ *exists and is not equal to zero. In addition,* (1) *if* $n+j$ *is odd,* $f^{(j)}(x) \geqslant \beta_n^j$ *and this inequality is best possible*; (2) *if* $n+j$ *is even,* $f^{(j)}(x) \leqslant \beta_n^j$ *and this inequality is best possible.*

PROOF. Suppose $n+j$ is odd and consider statement 1 of the theorem. Since the class F goes over into itself under translation to the left, it suffices to show $f^{(j)}(0) \geqslant B_{n,k}^{(j)}(0)$ for each $f \in F$. Suppose that this is not the case. So for some f we have $f^{(j)}(0) < B_{n,k}^{(j)}(0) < 0$. We fix α, $0 < \alpha < 1$, so that $B_{n,k}^{(j)}(0) - \alpha f^{(j)}(0) = 0$ and put $h(x) = B_{n,k}(x) - \alpha f(x)$ on the interval $[0, 2b^{-1}]$.

Since $\|\alpha f\| \leqslant \alpha < 1$, equioscillation of $B_{n,k}$ implies that h has $n+2k$ distinct zeros interior to the interval $[0, 2b^{-1}]$. Hence by Rolle's theorem, $h^{(j)}$ has $n-j+2k$ zeros in the interior of the interval. On taking account of $h(0) = 0$, we conclude that $h^{(j)}$ has $n-j+2k+1$ zeros, from which it follows that $h^{(n-1)}$ must have $2k+2$ zeros. This yields a contradiction, since from the hypothesis $f^{(n)}(x) \leqslant 1$ a.e., it follows that $h^{(n-1)}$ must consist of exactly $k+1$ increasing curves and as such can have no more than $2k+1$ zeros. (Here, of course, we count a jump discontinuity across the x-axis as a simple zero.)

Concerning the lack of a finite upper bound, one need merely consider the functions $f(x) = (-1)^{n+1}M^n(1/M - x)_+^n$, $M > 0$. Clearly, $f \in F$ no matter how large M, and if $n+j$ is odd, $f^{(j)}(0)$ is large with M. The proof of statement 2 is exactly the same.

Consideration of the above shows in addition that if n and j are fixed with $n+j$ odd then the sequence $\{B_{n,k}^{(j)}(0)\}_{k=0}^{\infty}$ is monotonically increasing and bounded above by 0; hence we can set $\beta_n^j = \lim_{k\to\infty} B_{n,k}^{(j)}(0)$. Similar remarks hold for $n+j$ even. In both cases one easily verifies that $\beta_n^j \neq 0$ and that the inequalities of the corollary are valid.

Now fix an interval $[0, A]$. Using the Ascoli-Arzela theorem and extracting subsequences if necessary, we define

$$B_n(x) = \lim_{k\to\infty} B_{n,k}(x).$$

When this is accomplished on each interval $[0, A]$, we then have a function $B_n(x)$ well defined on the positive axis and $B_n^{(j)}(0) = \beta_n^j$. $B_n(x)$ is not in F, although we can show that $B_n(x) \in C^{n-2}$. Nevertheless, by convolving $B_n(x)$ with, say, the Weierstrass kernel we obtain $\widetilde{B}_n \in F$ with values $\widetilde{B}_n^{(j)}(0)$ arbitrarily close to β_n^j. Thus our inequalities are best possible.

In addition to these results, one can also consider the class F_M defined by the requirement $-M \leqslant f^{(n)}(x) \leqslant 1$ a.e. on R^+. For such a class, there are best possible finite upper bounds and lower bounds for each $f^{(j)}$, $1 \leqslant j \leqslant n-1$. These results are handled in a fashion similar to the above. The main difference is that now there are two auxiliary functions $H_n(x)$ and $G_n(x)$ which together play the role of $B_n(x)$ and supply the best constants in our inequalities. As an example of these results, when $n = 2$ and $f \in F_M$, we have $-2\sqrt{2} \leqslant f'(x) \leqslant 2\sqrt{2M}$. Thus the lower bound on f'' governs the upper bound for f'. This feature persists for the higher derivatives as a dependence on the parity of $n + j$.

REFERENCES

1. A. S. Cavaretta, Jr., *An elementary proof of Kolmogorov's theorem,* Amer. Math. Monthly **81** (1974), 480–486. MR **49** #5269.

2. L. Hörmander, *A new proof and a generalizaation of an inequality of Bohr,* Math. Scand. **2** (1954), 33–45. MR **16**, 354.

3. R. S. Johnson, *On monosplines of least deviation,* Trans. Amer. Math. Soc. **96** (1960), 458–477. MR **23** #A270.

4. E. Landau, *Einige Ungleichungen für zweimal differentierbare Funktionen,* Proc. London Math. Soc. **13** (1913), 43–49.

5. I. J. Schoenberg,and A. S. Cavaretta, Jr., *Solution of Landau's problem concerning higher derivatives on the half-line,* Math. Research Center Technical Summary Report, No. 1050, University of Wisconsin, Madison, Wis., 1970.

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