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FIXED POINTS OF DISK ACTIONS

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As a sequel to a previous announcement [3], the author can now give a complete classification up to homotopy type of which spaces can occur as fixed point sets of smooth actions of a given compact Lie group on disks. The result is contained in Theorems 1 to 3 below. For a group G, G_0 denotes its identity component.

THEOREM 1. *Let G be a compact Lie group, and F a finite CW complex. Then there exists a smooth action of G on a disk with fixed point set having the homotopy type of F if and only if*:

1. $G \cong T^n$ $(n \geqslant 1)$: *F is $\mathbf{Z}$-acyclic*;
2. *G_0 a torus and* $|G/G_0| = p^a$ *(p prime, $a \geqslant 1$): F is $\mathbf{Z}_p$-acyclic,*
3. *G_0 not a torus or G/G_0 not of prime power order*: $\chi(F) \equiv 1 \pmod{n_G}$ *for some fixed integer n_G.*

In order to describe the calculations of n_G, some classes of finite groups are defined, as in [3] and [4]. $\mathcal{G}^1$ denotes the class of all G with normal subgroup P of prime power order, such that G/P is cyclic. For q prime, $\mathcal{G}^q$ denotes the class of all G with normal subgroup $H \in \mathcal{G}^1$ of q-power index. Then one gets

THEOREM 2. 1. *If G_0 is not a torus, then $n_G = 1$.*

2. *If G_0 is a torus, then $n_G = n_{G/G_0}$.*

3. *If G is finite, then $n_G = 0$ if and only if $G \in \mathcal{G}^1$; if $G \notin \mathcal{G}^1$ then for any prime q, $q \mid n_G$ if and only if $G \in \mathcal{G}^q$.*

In Theorem 1, the necessity of the conditions in (1) and (2) follow from standard Smith theory. Sufficiency follows in (2) from Jones [2], and in (1) is trivial ($G * F$ is contractible and can be thickened up to a disk action by Theorem 6 of [4]).

For finite G, the existence of n_G and the calculations in Theorem 2, part 3, were proven in [4]. Furthermore, if G_0 is a torus and $G \supsetneqq G_0$, then F clearly has the homotopy type of the fixed point set of a disk action of G if and only if it does the same for G/G_0, so $n_G = n_{G/G_0}$. The case where G_0 is nontoral will be dealt with below; the above theorems say that *any* finite homotopy type can occur as fixed point set for such G.

The following result, completing the calculation of n_G, was obtained in

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Theorem 4 of [5] by studying the projective obstruction $\gamma_G(F)$ first introduced in [4].

THEOREM 3. *For any finite group G, $n_G = 4$ if and only if*:

1. *G is a semidirect product* $0 \longrightarrow \mathbf{Z}_n \longrightarrow G \longrightarrow \mathbf{Z}_{2^k} \longrightarrow 0$ (*n odd*) *given by an automorphism* $\alpha \in \mathrm{Aut}(\mathbf{Z}_n)$.
2. $G \notin \mathcal{G}^1$, *but the subgroup of index* 2 *is in* $\mathcal{G}^1$.
3. *Letting* α *also denote the induced automorphism of* $\mathbf{Z}\zeta_n$ (*the ring generated by the nth roots of unity*), *there is no unit* $u \in (\mathbf{Z}\zeta_n)^*$ *such that* $\alpha(u) = -u$.

Otherwise, n_G *equals* 0, 1 *or a product of distinct primes.*

Groups fulfilling conditions 1–3 *do* actually exist, the smallest being given by $\langle a, b: a^{15} = b^4 = e, bab^{-1} = a^2\rangle$.

It remains to describe the case of groups with nontoral identity component; by Bredon's construction [1, §I.8] it is enough to construct a fixed point free action of any such group on a disk. The following theorem provides some very specific examples of such actions. The concept of a *family* of subgroups is used, as defined by tom Dieck.

THEOREM 4. *Let G be a compact Lie group, and* $\mathcal{F}$ *a nonempty family of subgroups. Then there exists a smooth action of G on a disk D such that* D^H *is a disk for* $H \in \mathcal{F}$ *and empty for* $H \notin \mathcal{F}$, *if and only if*:

1. *For any pair of subgroups* $H \lhd K$ *in G, for which K/H has prime order, either both H and K are in* $\mathcal{F}$ *or neither is.*
2. $\mathcal{F}$ *is closed in the space of closed subgroups of G with the Hausdorff topology.*

In particular, the family of subgroups H such that H_0 is a torus and H/H_0 solvable meets these conditions.

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