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THE TOTAL CURVATURE OF KNOTTED SPHERES

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Chern and Lashof [1] proved several inequalities concerning the total curvature of an immersed manifold. Their second result is a weak generalization of the Fary-Milnor theorem [2], [5] for closed space curves. In this paper, a stronger result (Corollary 1), the complete homotopy extension, is stated and proved. I would like to thank Bill Pohl for conversations surrounding the formulation and proof of this result.

I. **Background.** Let $x: M^n \longrightarrow E^{n+N}$ be a C^∞-immersion into Euclidean space of dimension $n + N$ $(N > 0)$; and B_ν be the bundle of unit normal vectors of $x(M^n)$. A point of B_ν is a pair $(p, \nu(p))$, where $\nu(p)$ is a unit normal vector to $x(M^n)$ at $x(p)$. The map $\bar{\nu}: B_\nu \longrightarrow S_0^{n+N-1}$, into the unit sphere of E^{n+N}, is defined by $\bar{\nu}(p, \nu(p)) = \nu(p)$.

The Lipschitz-Killing curvature [1], $G(p, \nu)$ at $\nu(p)$, is then given by the $\bar{\nu}$-ratio of corresponding volume elements in S_0^{n+N-1} and B_ν. The *total curvature of M^n at p* is $K^*(p) = \int |G(p, \nu)|\, d\sigma$, the integral being taken over the sphere of unit normal vectors at $x(p)$. The *total curvature of M^n* is given by $K^* = K^*(M) = \int_{p \in M} K^*(p)\, dV$.

The first two Chern-Lashof theorems can be stated as: Given M^n compact without boundary, and $c(m)$ the area of the unit hypersphere $S_0^m \subset E^{m+1}$, then:

COROLLARY 1. $K^*(M) \geqslant 2c(n + N - 1)$.

COROLLARY 2. *If $K^*(M) < 3c(n + N - 1)$, then M is homeomorphic to S^n.*

The essential argument of their proof can be summarized as a lemma.

LEMMA 1. *If, for almost all $v_0 \in S_0^{n+N-1}$, the height function $\langle v_0, - \rangle$: $x(M) \longrightarrow R$ has at least k distinct critical points, then $K^*(M) \geqslant kc(n + N - 1)$.*

Their method is an adaptation of the technique used by Fenchel [3]. This fact suggested that Corollary 2 is a weak generalization of Fary-Milnor.

II. **The main result.** In this section, a curvature inequality is given which distinguishes between different knottings of S^n. The method, based on Chern-Lashof, takes off from a remark of Fox [4] in which P. L. approximations yield the corresponding S^1 result.

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For simplicity of presentation, attention is restricted to knotted spheres; that is, $M^n = S^n$ and codimension $N = 2$. Recall, for a mapping $x\colon S^n \longrightarrow E^{n+2}$, the group of the map is $\pi(x) = \pi_1 [E^{n+2} - x(S^n)]$.

DEFINITION 1. $g(x) =$ the minimal number of generators needed to present $\pi(x)$.

THEOREM I. $K^*(S^n) \geqslant 2g(x)c(n + 1)$.

COROLLARY 1. *If* $K^*(S^n) < 4c(n + 1)$, *then* $\pi(x) = Z$.

The corollary follows trivially since any $\pi(x)$ has Z as a subgroup. Theorem I is a consequence of Lemma 1 combined with the obvious.

PROPOSITION 1. *For almost all* $v_0 \in S_0^{n+1}$, *the height function* $\langle v_0, -\rangle\colon x(S^n) \longrightarrow R$ *has at least* $2g(x)$ *distinct critical points.*

PROOF. Since we only need to account for an open dense subset of the v_0's, fix a height $\langle v_0, -\rangle$ which is Morse. Choose a basepoint, *, which is "higher" than $x(S^n)$. The proposition is shown by constructing a canonical set of generators for $\pi(x, *)$, and deforming an arbitrary loop, $\gamma \in \pi(x, *)$, into a sum of these. The deformation is first described. The required generating set will be obvious at the outcome.

Since * is higher than $x(S^n)$, assume that the loop γ is strictly lower than *. Now, define a *lifting-homotopy* as a homotopy $H(x, t)$ which is always moving to higher levels, that is one where $\langle v_0, H(x, t)\rangle$ is nondecreasing in t for all fixed x in the loop parametrization. The problem involved is to determine the obstructions in $x(S^n)$ preventing γ from being pulled up all the way. Clearly, any such phenomenon will be local. The crucial observation is that γ can only be "caught" on maximums of $\langle v_0, -\rangle\colon x(S^n) \longrightarrow R$.

Take a collection of open collared balls, $U_i \subset W_i$, in E^{n+2} such that: (1) $\{U_i\}$ is a finite covering of a simply-connected volume enclosing $x(S^n)$; (2) each critical point p is contained in only one ball W_i; and (3) there are Morse-coordinates for $(W_i \cap x(S^n))$ whose axes are strictly monotonic w.r.t $\langle v_0, -\rangle$. Clearly, any part of γ lying in a U_i not containing a critical point can be lifted out of the ball. This means that attention can be focused on the $U_1, \ldots, U_k$ containing $p_1, \ldots, p_k$.

Now, suppose that p_j is not a maximum. Then the height function is increasing on at least one Morse-axis, and the piece of $x(S^n)$ locally obstructing γ has at least codim 3. There are index$(p_j) > 0$ degrees of freedom with which to translate a segment of γ and lift it into the collar $(W_j - U_j)$ such that it lies above $U_j \cap x(S^n)$. After a finite number of such movements, γ will only be obstructed by balls containing maximums.

Next, assign a unique 'canonical' element of $\pi(x)$ to each maximum. For p_j a maximum, fix a loop γ_j which passes under p_j only once. This can be arranged (inside W_j) by adding a lower hemisphere to $U_j \cap x(S^n)$, and taking γ_j as a generator which leaves W_j through the north pole and is increasing till $*$. Any segments of γ stuck in U_j can be lined up (inside W_j) with γ_j. The rest of the loop goes up and away. Hence, the collection $\{\gamma_j\}$ is a set of generators for $\pi(x)$.

Summarizing, any $\langle v_0, -\rangle$ has at least $g(x)$ maximums. Next, if C_i = the number of critical points of index i, then the Morse equality gives: (1) for n odd, $\Sigma_{i=1}^{n}(-1)^{i+1}C_i = C_0 \geqslant g(x)$, and there are at least $g(x)$ critical points other than maximums; (2) for n even, there is at least one minimum, hence: $\Sigma_{i=1}^{n-1}(-1)^{i+1}C_i = C_0 + C_n - 2$, and there are at least $(g(x) - 1)$ critical points other than maximums and minimums. In either case, the proof is complete.

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