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UNIQUE FACTORIZATION IN RANDOM VARIABLES

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The problem of determining the potential $q(x)$ from "spectral data" for the equation

$$-y''(x) + q(x)y(x) = \lambda y(x), \qquad -\infty < x < \infty,$$

has been studied extensively. For a review, see [1] and [5].

A typical kind of result, associated with the names of Gelfand and Levitan tells you that if the discrete spectrum, the normalizing constants, and the reflection coefficient $R(k)$ are known, then $q(x)$ can, in principle, be determined uniquely.

Here we consider a random version of this problem. We envisage a situation where one keeps records of "spectral data" for noisy versions of the potential $q(x)$ and attempts to determine the "mean potential" from the distribution of the data.

It turns out that in this random case a smaller set of quantitites–measured over and over again–give a lot of information about $q(x)$. A similar situation develops in a variety of different setups (see, for instance, [3] and [4]).

Let $-\nabla^2$ stand for the $n \times n$ matrix

$$-\nabla^2 \equiv \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & 0 \\ & -1 & 2 & & \\ & 0 & & \ddots & -1 \\ & & & -1 & 2 \end{pmatrix}.$$

THEOREM I. *Let $q_1, \ldots, q_n$ be independent Gaussian random variables with unknown means $\overline{q}_1, \ldots, \overline{q}_n$ and variances all equal to one. Then the joint distribution function of* $\mathrm{tr}(-\nabla^2 + qI)^k$, *$k = 1, \ldots, n$, determines the vector $\overline{q}_1, \ldots, \overline{q}_n$ up to a global reflection $\overline{q}'_i \equiv \overline{q}_{n-i+1}$.*

The theorem above says that the spectrum determines the potential up to a trivial reflection. This is to be compared with the nonrandom case where, in

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general, $n!$ choices of potential are compatible with a given spectrum. For results of this kind, see [2].

The last remark indicates that one can guess how does the typical situation look by "ignoring" the off diagonal elements in $-\nabla^2 + qI$. This gives a link between the two theorems below.

MAIN THEOREM. *Let $X_1, X_2, \ldots, X_n$ be independent Gaussian random variables with variance one and unknown means $a_1, \ldots, a_n$. Then the distribution function of the product $Z = X_1 \cdots X_n$ suffices to determine the product $a_1 a_2 \cdots a_n$ and the quantities $a_1^2, a_2^2, \cdots, a_n^2$ up to order.*

THEOREM II. *Under the conditions of Theorem* I, *the distribution function of* $\det(-\nabla^2 + qI)$ *determines $\bar{q}$ up to an even number of changes of sign and a global reflection.*

We give below a proof of the main theorem, which gives its title to this note, in the first nontrivial case, i.e., $n = 4$. First, some remarks.

I. The Gaussian character of the X_i's is only a convenient device to get an easier proof. Actually if the $(X_i - a_i)$'s have a symmetric distribution with second moment μ_2 and fourth moment μ_4, a condition like $\mu_2/\sqrt{2} > 3\mu_2^2 - \mu_4$ suffices to make the proof below work. In the Gaussian case the right-hand side vanishes and our conditions say that we are in the truly random case $\mu_2 > 0$.

II. There is no need to assume that the distribution functions of the $(X_i - a_i)$'s are the same. It is also unnecessary to assume these distributions are known in advance. In this way one gets

COROLLARY. *From measurements of the volume of (slightly defective) replicas of a parallelogram, one can infer its length, width and height up to order.*

PROOF OF THE MAIN THEOREM ($n = 4$). For simplicity, take all the means $a_1, \ldots, a_4$ to be nonzero. Introduce the elementary symmetric functions in the unknowns $a_1^2, a_2^2, a_3^2, a_4^2$, i.e.,

$$\sigma_1 = a_1^2 + a_2^2 + a_3^2 + a_4^2, \qquad \sigma_2 = \sum_{i<j} a_i^2 a_j^2,$$

$$\sigma_3 = \sum_{i<j<k} a_i^2 a_j^2 a_k^2, \qquad \sigma_4 = a_1^2 a_2^2 a_3^2 a_4^2.$$

The first three moments of the random variable Z give

$$Z_1 = a_1 a_2 a_3 a_4 = \sigma_4^{1/2}, \qquad Z_2 = \sigma_4 + \sigma_3 + \sigma_2 + \sigma_1 + 1,$$

$$Z_3 = (\sigma_4 + 3\sigma_3 + 3^2\sigma_2 + 3^3\sigma_1 + 3^4)Z_1.$$

The fourth moment Z_4 is a quadratic function of the σ_i's which is best written in terms of the new variables

$$x = \frac{1}{\sqrt{38}}(3\sigma_1 - 18\sigma_2 + \sigma_3), \quad y = \frac{1}{\sqrt{19}}(9\sigma_1 + 3\sigma_2 + 3\sigma_3),$$

$$z = \frac{1}{\sqrt{2}}(3\sigma_1 - \sigma_3).$$

We have

$$Z_4 = 19y^2 - 12z^2 + y(C_1\sigma_4 + C_2) + z(C_3\sigma_4 + C_4) + \sigma_4^2 + C_5\sigma_4$$

with $C_1, C_2 > 0$.

Now from Z_1, Z_2 and Z_3, one can read off both σ_4 and z. Then Z_4 leads to two possible choices of y, only one of which can be positive since C_1 and C_2 are positive. But y is positive by definition and thus can be determined from Z_4 and then used along with z to get x from Z_2. We have thus obtained σ_1, σ_2, σ_3, σ_4 from Z_1, Z_2, Z_3, Z_4 and the proof is finished.

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