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THE GROUP ALGEBRA OF A TORSION-FREE ONE-RELATOR GROUP CAN BE EMBEDDED IN A FIELD

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Let G_1, G_2 be groups, w a (skew) field, and H a common subgroup of G_1 and G_2. Suppose that the group algebra wG_i is embedded in a field $\overline{G}_i$ and suppose that the fields $\overline{H}_1$ and $\overline{H}_2$ generated by wH in $\overline{G}_1$ and in $\overline{G}_2$ are wH-isomorphic. In order that the group algebra $w(G_1 \underset{H}{*} G_2)$ be embedded in $\overline{G}_1 \underset{\overline{H}_1=\overline{H}_2}{*} \overline{G}_2$ ($*$ denotes coproduct (= free product with amalgamation) of groups and rings), it is sufficient that $\overline{H}_i$ and wG_i be *linearly disjoint* over wH in $\overline{G}_i$, i.e., that the multiplication map $\overline{H}_i \otimes_{wH} wG_i \longrightarrow \overline{G}_i$ be injective.

If R is a semifir, then [3, Chapter 7] there is a field $U(R)$, the universal field of fractions of R, embedding R such that each automorphism of R extends to an automorphism of $U(R)$. If F is a free group, wF is a semifir and has a universal field of fractions.

THEOREM 1. *Let $G = \mathrm{gp}\langle t, x, y, \ldots, z; R(t, x, y, \ldots, z)\rangle$ be a torsion-free one-relator group and w a field. Then wG can be embedded in a field $\overline{G}$ with the following property: If $S \subset \{t, x, y, \ldots, z\}$ and S omits at least one letter involved in R, then the subfield $\overline{\mathrm{gp}(S)}$ of $\overline{G}$ generated by the free group algebra $w(\mathrm{gp}(S))$ is its universal field of fractions. Further, $\overline{\mathrm{gp}(S)}$ and wG are linearly disjoint over $w(\mathrm{gp}(S))$ in $\overline{G}$.*

SKETCH OF PROOF. Let the *complexity* of R (which we assume is cyclically reduced) be the length of R minus the number of letters involved in R. The proof is by induction on the complexity of R. (If R has complexity zero, then G is free and $\overline{G} = U(wG)$ has the required properties [5].)

CASE 1. R has exponent sum zero on some letter, say t. By the proof of the Freiheitssatz [7, §4.4], the normal closure N of $\{x, y, \ldots, z\}$ in G is a tree product

$$\cdots * N_{-1} \underset{A_{-1,0}}{*} N_0 \underset{A_{0,1}}{*} N_1 * \cdots$$

where N_0 is a one-relator group whose defining relator is less complex than R, $A_{0,1}$ is a free group generated by a proper subset of the generators of N_0, and

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$N_i = t^{-i}N_0t^i, A_{i,i+1} = t^{-i}A_{0,1}t^i$.

To prove the theorem, we copy this construction with fields. By induction, wN_0 is embedded in a field $\bar{N}_0$. Choosing copies $\bar{N}_i$ of $\bar{N}_0$ (with the isomorphism $\bar{N}_0 \longrightarrow \bar{N}_i$ extending $N_0 \longrightarrow N_i$) and defining $\bar{A}_{i,i+1}$ accordingly, we form the tree of rings

$$\cdots * \bar{N}_{-1} \underset{\bar{A}_{-1,0}}{*} \bar{N}_0 \underset{\bar{A}_{0,1}}{*} \bar{N}_1 * \cdots .$$

The colimit R of this system (= tree product) is [2] a semifir with a universal field of fractions $\bar{N}$ which we can show embeds wN.

We now extend, via the automorphism induced by conjugation by t, $\bar{N}$ to a field $\bar{G}$ which embeds wG.

To complete the proof of the exponent sum zero case, we show that each of the usual expressions of N as a tree product induces a corresponding tree product structure inside $\bar{N}$.

CASE 2. Changing notation slightly, assume now that

$$G = \text{gp}\langle a, b, \ldots, z; R'(a, b, \ldots, z)\rangle$$

and that R' has nonzero exponent sum on every letter. Let $H = G * \text{gp}\langle t\rangle$. Two letters, say a and b, and integers k, l can be found such that if we set $x = at^k$, $y = bt^l$, and $R(t, x, y, \ldots, z) = R'(xt^{-k}, yt^{-l}, \ldots, z)$, then

$$H = \text{gp}\langle t, x, y, \ldots, z; R(t, x, \ldots, z)\rangle$$

is such that R has exponent sum zero on t and such that N_0, obtained as in Case 1, has a defining relator less complex than R'. We then work with H and find that the subfield of $\bar{H}$ generated by wG is a field with the required properties. □

COROLLARY. *If H is any group such that wH can be embedded in a field $\bar{H}$, then $w(H \times G)$ can be embedded in a field (by choosing w to be $\bar{H}$ in Theorem 1).* □

COROLLARY. *If A is a commutative domain, then AG is regular, i.e., $AG \otimes_A AG$ is a domain.* □

Now write $G = F/P$, where F is a free group and P is the normal closure of the element R. let $\mathfrak{p}$ be the augmentation ideal of P in AF. From the universal derivation sequence,

$$0 \longrightarrow \mathfrak{p}/\mathfrak{p}^2 \longrightarrow T(AF; AG, AG) = \Omega \longrightarrow AG \otimes_A AG \longrightarrow AG \longrightarrow 0,$$

[4], [1] we obtain

THEOREM 2. *If $A = \mathbf{Z}$, then $\mathfrak{p}/\mathfrak{p}^2$ is a free $\mathbf{Z}G$-bimodule (cf. [6]).*
If A is a PID, then the global dimension of AG is at most 3.
If A is a field, then the global dimension of AG is at most 2. □

Proofs will appear in Communications in Algebra.

ADDED IN PROOF. The global dimension statements of Theorem 2 also follow from Corollary 4.2 of F. Waldhausen's *K-theory of generalized free products* (mimeographed).

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