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THE SELBERG TRACE FORMULA FOR CONGRUENCE SUBGROUPS

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1. **Introduction.** The Selberg trace formula for SL(2, **R**) is commonly understood to be a non-Abelian analog of the Poisson summation formula. The formula arises from letting a Fuchsian group Γ act on the upper half-plane H and contains four basic contributions: identity, hyperbolic, elliptic, and parabolic [2, pp. 95–108], [3, pp. 72–79]. Because of its possible number-theoretic applications, it seems only natural to calculate the trace formula explicitly for various congruence subgroups of SL(2, **Z**) and see what happens.

From the general theory, one knows that the parabolic (or arithmetic) contribution will be Tr($\mathcal{M}$), where

$$\mathcal{M} = \frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\Phi'(s)\Phi(s)^{-1}\,dr + \tfrac{1}{4}[I - \Phi(\tfrac{1}{2})]h(0) - \left[g(0)\ln 2 + \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\frac{\Gamma'}{\Gamma}(1+ir)\,dr\right]I.$$

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We use here the notation of [2, p. 108] and write $s = ½ + ir$. $\Phi(s)$ is the so-called Eisenstein matrix. M will therefore be a $C \times C$ matrix, where $C =$ the number of inequivalent cusps. For the other three contributions, see [2, pp. 95–108].

The groups Γ we want to consider are::SL(2, Z), $\Gamma_0(N)$, $\Gamma_1(N)$, $\Gamma_2(N)$. These last three groups are defined by the congruence conditions:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, \pm \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N.$$

We will assume that $N \geqslant 3$ is square-free. In order to evaluate the parabolic contribution for each group, one must first calculate $\Phi(s)$ and then $\mathrm{Tr}(M)$. The necessary computations are quite lengthy (for large N).

2. **The trace formulas.** Because we are interested only in the "arithmetic" part of the formulas, we need only give $\mathrm{Tr}(M)$. We introduce the number-theoretic functions $\phi(n)$, $\Lambda(n)$, $\omega(n)$ as in [1, pp. 233, 253, 354] and let $\epsilon = \pm 1$.

Case SL(2, Z). $C = 1$:

$$\mathrm{Tr}(M) = g(0) \ln (\pi/2) - \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\left\{\frac{\Gamma'}{\Gamma}(½ + ir) + \frac{\Gamma'}{\Gamma}(1 + ir)\right\} dr + 2 \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n} g(2 \ln n).$$

Case $\Gamma_0(N)$. $C = 2^{\omega(N)}$:

$$\mathrm{Tr}(M) = C\,\mathrm{Tr}(M_{\mathrm{SL}(2,\mathbf{Z})}) - Cg(0) \ln N - C \sum_{p|N} \sum_{n=p^r} \frac{\Lambda(n)}{n} g(2 \ln n).$$

Case $\Gamma_1(N)$. $C = ½\phi(N)2^{\omega(N)}$:

$$\mathrm{Tr}(M) = (1/8)h(0)2^{\omega(N)}[\phi(N) - 2] + Cg(0)[\ln(\pi/2) - (3/2) \ln N]$$
$$- \frac{C}{2\pi}\int_{-\infty}^{\infty} h(r)\left\{\frac{\Gamma'}{\Gamma}(½ + ir) + \frac{\Gamma'}{\Gamma}(1 + ir)\right\} dr$$
$$+ C\left[2 \sum_{n \equiv \epsilon \bmod N} \frac{\Lambda(n)}{n} g(2 \ln n)\right]$$
$$+ 2^{\omega(N)} \sum_{p|N} F\left(\frac{N}{p}\right)\left[\frac{1}{2} g(0) \ln p + \sum_{\substack{n \equiv \epsilon \bmod (N/p); \\ n=p^r}} \frac{\Lambda(n)}{n} g(2 \ln n)\right]$$

where $F(k) = ½\phi(k)$ when $k \geqslant 2$, and $F(1) = 1$.

Case $\Gamma_2(N)$. $C = \frac{1}{2}\Pi_{p|N}(p^2 - 1)$:

$$\mathrm{Tr}(M) = (1/8)h(0) \prod_{p|N} (p+1) \cdot [\phi(N) - 2] + Cg(0)[\ln(\pi/2) - 2\ln N]$$

$$- \frac{C}{2\pi} \int_{-\infty}^{\infty} h(r) \left\{ \frac{\Gamma'}{\Gamma}(\tfrac{1}{2} + ir) + \frac{\Gamma'}{\Gamma}(1 + ir) \right\} dr$$

$$+ C \left[2 \sum_{n \equiv \epsilon \bmod N} \frac{\Delta(n)}{n} g(2 \ln n) \right]$$

$$+ \prod_{p|N} (p+1) \cdot \left[g(0) \sum_{p|N} F\left(\frac{N}{p}\right) \frac{\ln p}{p+1} \right.$$

$$\left. + 2 \sum_{p|N} \sum_{\substack{n \equiv \epsilon \bmod (N/p); \\ n = p^r}} \frac{\Delta(n)}{n(p+1)} F\left(\frac{N}{p}\right) g(2 \ln n) \right].$$

Due to limitations of space, we will omit the formulation of the $\Phi(s)$ matrices. The case $\Gamma = \mathrm{SL}(2, \mathbf{Z})$ can be found in [2, p. 46].

3. **Connections with analytic number theory.** Trace formulas are important for several reasons. One reason is that they exhibit a very striking structural similarity to certain explicit formulas of prime number theory (especially Weil [5]). Thus, in the notation of [2], we have

$$\sum_{\gamma} h(\gamma) = h\left(\frac{i}{2}\right) + h\left(-\frac{i}{2}\right) - g(0) \ln \pi + \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) \frac{\Gamma'}{\Gamma} \left(\frac{1}{4} + \frac{1}{2} ir\right) dr$$

$$- 2 \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} g(\ln n),$$

where $\frac{1}{2} + i\gamma$ are the nontrivial zeros of $\zeta(s)$ (of course $\gamma \in \mathbf{C}$). One might possibly hope to interpret Weil's formula as actually being a trace formula (or else limit thereof). This would provide some important insight into the Riemann hypothesis.

If the formulas of §2 offer any clue at all, we see that there will be several important obstructions:

(a) $2 \ln n$ and n appear in place of $\ln n$ and $\sqrt{n}$;

(b) the $\Lambda(n)$ terms appear with a "+" sign instead of a "–";

(c) the identity and hyperbolic contributions to the trace formula significantly "overshadow" the parabolic (in magnitude).

4. **Hecke operators.** As mentioned in [3], [4], one can also develop a trace formula for $\mathrm{Tr}(ML)$, where M is a modular correspondence and L is the

usual integral operator $Lf(z) = \int_H k(z, w)f(w)dw$. The case of $\Gamma = SL(2, \mathbf{Z})$ and $M = T_p$ (the usual Hecke operator) can be computed explicitly. The only serious difficulty arises from matrices $\tau = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in GL(2, \mathbf{Z})$ having $AD - BC = p$ and $|A + D| = p + 1$. The corresponding contribution to $\mathrm{Tr}(T_pL)$ is

$$g(\ln p)[p^{1/2} \ln p + 2p^{1/2} \ln \pi + 2p^{1/2} \ln(p^{1/2} - p^{1/2})] - \frac{2g(\ln p)}{p^{1/2} - p^{-1/2}} \sum_{k=1}^{p-1} \ln(k, p-1)$$

$$- \tfrac{1}{2}p^{1/2}h(0) + 2p^{1/2} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n} [g(2 \ln n - \ln p) + g(2 \ln n + \ln p)]$$

$$+ p^{1/2} \int_{\ln p}^{\infty} g(u) \frac{e^u + 1}{(e^{u/2} + p^{1/2})(e^{u/2} - p^{-1/2})} du$$

$$- \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r)[p^{1/2+ir} + p^{1/2-ir}] \frac{\Gamma'}{\Gamma}(\tfrac{1}{2} + ir)dr.$$

One finds here a certain resemblance to the formulas of §2. As a result, the obstructions (a)–(c) still seem to apply.

5. Detailed proofs will be published elsewhere.

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