BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 4, July 1975

GENERIC PROPERTIES OF RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS[1]

BY JOHN MALLET-PARET

Communicated by Hans Weinberger, February 18, 1975

Consider the retarded functional differential equation (RFDE)

$$\dot{x}(t) = f(x_t) \tag{1}$$

where as in [1], $x_t(\theta) = x(t + \theta), -1 \leqslant \theta \leqslant 0, x_t \in C = C([-1, 0], R^n)$, and $f \in X = C^\infty(C, R^n)$. Oliva [5] showed fixed points of (1) generically are hyperbolic; here we generalize to the theorem of Kupka [2], Markus [3] and Smale [7]. With an appropriate Whitney (Baire) topology on X, we have

THEOREM 1. *The set of $f \in X$ for which*

1. *all fixed points and all periodic solutions of* (1) *are hyperbolic,*
2. *all global unstable manifolds are injectively immersed in C, and*
3. *all global unstable and local stable manifolds intersect transversally*

is a residual subset of X.

The restriction to the local stable manifold in 3. is necessary as we lack backwards existence and uniqueness.

If we consider only equations

$$\begin{gathered}\dot{x}(t) = F(x(t - \tau_1), x(t - \tau_2), \ldots, x(t - \tau_p)),\\ 0 \leqslant \tau_1 < \tau_2 < \cdots < \tau_p \leqslant 1 \text{ fixed,}\end{gathered} \tag{2}$$

with $F \in C^\infty(R^{np}, R^n)$, we obtain

THEOREM 2. *Let $\tau_1 = 0$. Then the set of $F \in C^\infty(R^{np}, R^n)$ for which* 1., 2. *and* 3. *above hold is a residual set.*

The question of what happens when $\tau_1 > 0$ seems to be open.

AMS (MOS) subject classifications (1970). Primary 34K15; Secondary 58F20.

Key words and phrases. Delay differential equation, functional differential equation, generic, hyperbolic periodic orbit, stable (unstable) manifold, transversality.

[1]This work was partially supported by National Science Foundation Grant GP 38955 at the University of Minnesota, and by National Science Foundation Grant GP 28931X2 at Brown University.

To prove Theorem 1, the various perturbations are constructed using the map $\delta_N: C \longrightarrow R^{n(N+1)}$

$$\delta_N\phi = (\phi(0), \phi(-1/N), \phi(-2/N), \ldots, \phi(-1)).$$

The next lemma, which is easily proved, is used when M is a periodic orbit or part of an unstable manifold.

LEMMA 1. *Let $M \subseteq C$ be an embedded C^∞ finite dimensional submanifold, and $K \subseteq M$ compact. Then there is a neighborhood $K \subseteq V \subseteq M$ in M such that for large N, $\delta_N: V \longrightarrow R^{n(N+1)}$ embeds V into $R^{n(N+1)}$ and is an immersion (and hence a diffeomorphism).*

We cannot use δ_N in Theorem 2 as all perturbations must take place in R^{np}. However, after approximating F with an analytic function, the following is used, with $x(t)$ periodic or lying on an unstable manifold.

LEMMA 2. *Suppose $\tau_1 = 0$ and let $x(t)$ be a solution of (2) on $(-\infty, 0]$, where F is analytic and x is bounded on $(-\infty, 0]$. Set $y(t) = (x(t-\tau_1), x(t-\tau_2), \ldots, x(t-\tau_p))$. Then*

1. *if x has least period $T > 0$, then with the exception of finitely many points in $[0, T]$, y is one-to-one on this interval;*
2. *if x is not periodic or constant, and $[a, b] \subseteq (-\infty, 0]$, then the same conclusion about y holds on $[a, b]$.*

PROOF. By a theorem of Nussbaum [4], $x(t)$ is analytic, so any self-intersection of $t \longrightarrow y(t)$ in R^{np} is either isolated or forms an analytic arc. In the latter case

$$y(t) = y(\sigma(t)) \quad \text{so } x(t) = x(\sigma(t)) \tag{3}$$

for some analytic σ defined in a t-interval I, with $\dot\sigma(t) \neq 0$ and $\sigma(t) \neq t$. Thus

$$\dot x(t) = F(y(t)) = F(y(\sigma(t))) = \dot x(\sigma(t)).$$

But differentiating (3) gives $\dot x(t) = \dot x(\sigma(t))\dot\sigma(t)$, so $\dot\sigma(t) \equiv 1$. Hence for some A, $x(t) = x(t + A)$ in I and thus for all t by analyticity. Thus 1 holds with A a multiple of T, proving the lemma.

REFERENCES

1. J. K. Hale, *Functional differential equations*, Applied Mathematical Sciences, vol. 3, Springer-Verlag, New York, 1971.

2. I. Kupka, *Contribution à la théorie des champs génériques*, Contributions to Differential Equations 2 (1963), 457–484; Addendum and corrections, ibid. 3 (1964), 411–420. MR 29 #2818a; #6134.

3. L. Markus, *Generic properties of differential equations*, Internat. Sympos. on Nonlinear Differential Equations and Nonlinear Mechanics, Academic Press, New York, 1963.

4. R. D. Nussbaum, *Periodic solutions of analytic functional differential equations are analytic*, Michigan Math. J. 20 (1973), 249–255. MR 48 #677.

5. W. M. Oliva, *Functional differential equations on compact manifolds and an approximation theorem*, J. Differential Equations 5 (1969), 483–496. MR 38 #5247.

6. M. M. Peixoto, *On an approximation theorem of Kupka and Smale*, J. Differential Equations 3 (1966), 214–227. MR 35 #499.

7. S. Smale, *Stable manifolds for differential equations and diffeomorphisms*, Ann. Scuola Norm. Sup. Pisa (3) 17 (1963), 97–116. MR 29 #2818b.

LEFSCHETZ CENTER FOR DYNAMICAL SYSTEMS, DIVISION OF APPLIED MATHEMATICS, BROWN UNIVERSITY, PROVIDENCE, RHODE ISLAND 02912

BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 4, July 1975

THE SELBERG TRACE FORMULA FOR CONGRUENCE SUBGROUPS

BY DENNIS A. HEJHAL

Communicated by S. Eilenberg, March 13, 1975

1. **Introduction.** The Selberg trace formula for SL(2, **R**) is commonly understood to be a non-Abelian analog of the Poisson summation formula. The formula arises from letting a Fuchsian group Γ act on the upper half-plane H and contains four basic contributions: identity, hyperbolic, elliptic, and parabolic [2, pp. 95–108], [3, pp. 72–79]. Because of its possible number-theoretic applications, it seems only natural to calculate the trace formula explicitly for various congruence subgroups of SL(2, **Z**) and see what happens.

From the general theory, one knows that the parabolic (or arithmetic) contribution will be $\mathrm{Tr}(\mathcal{M})$, where

$$\mathcal{M} = \frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\Phi'(s)\Phi(s)^{-1}\,dr + \tfrac{1}{4}[I - \Phi(\tfrac{1}{2})]h(0) - \left[g(0)\ln 2 + \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\frac{\Gamma'}{\Gamma}(1+ir)\,dr\right]I.$$

AMS (MOS) subject classifications (1970). Primary 10D10, 10H05.