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TOPOLOGICALLY DEFINED CLASSES OF GOING-DOWN DOMAINS

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1. **Introduction.** This note announces some results which build upon the studies of Dobbs [3], [4] and Dobbs and Papick [5] on going-down extensions and going-down domains. Whereas much of [4] was motivated by flatness (cf. [11, 5.D], [15]), the present work has a topological stimulus (cf. [7], [8, Proposition 1.10.13(a), (b′)], [10, pp. 145–160], [12], [14, Corollaire 2, p. 42]). We introduce and study new topologically defined classes of going-down domains, by considering how various going-down conditions on a domain R and its overrings relate to conditions on the topological space Spec(R).

Details, as well as a systematic study of the behavior of various classes of going-down domains under homomorphic images, localization and globalization, integral change of rings, and the "$D + M$ construction", will appear elsewhere.

2. **Notation.** Let $\mathcal{P}$ (respectively, $\mathcal{Q}$) be a property which may be satisfied by an extension of (commutative integral) domains (respectively, by the map induced on prime spectra by an extension of domains). A domain R is a *$\mathcal{P}$ domain* (*respectively, $\mathcal{Q}$ domain*) if $R \subset T$ (respectively, Spec$(T) \longrightarrow$ Spec(R)) satisfies $\mathcal{P}$ (respectively, $\mathcal{Q}$) for each overring T of R.

3. **Going-down domains and *i*-domains.** In this section, we introduce tools needed for the remaining sections, and at the same time extend and clarify notions already present in the literature. Recall from [4] and [5] that a domain R is called a *going-down domain* (written R is GD) in case we take $\mathcal{P}$ = GD; and R is said to be *treed* if Spec(R), as a partially ordered set under inclusion, is a tree. In [4], it is shown that a GD domain must be treed; an example of Lewis, described in [13], shows that the converse need not be true. By taking $\mathcal{P}$ = mated (as defined by Dawson and Dobbs [2]) and $\mathcal{Q}$ = injec-

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[1] This note summarizes a portion of the author's Ph. D. thesis written under the direction of Professor David E. Dobbs at Rutgers University. The author wishes to thank Professor Dobbs for this continual assistance and patient guidance in the preparation of his thesis.

tive, we use a corollary of Zariski's main theorem [14, Corollaire 2, p. 42] to show

PROPOSITION 3.1. *R is a mated domain* $\Longleftrightarrow$ *R is an i(njective)-domain.*

COROLLARY 3.2. *Every overring of an i-domain is GD.*

Observe that R is an i-domain if and only if R_M is an i-domain for each maximal ideal M of R. The local case is summarized next. Note first that Brase [1] and Dobbs [6] independently considered the condition that $\bar{R}$, the integral closure of R, be valuation.

PROPOSITION 3.3. *R is a local i-domain* $\Longleftrightarrow$ *$\bar{R}$ is a valuation ring* $\Longleftrightarrow$ *each overring of R is local.*

THEOREM 3.4. *R is an i-domain* $\Longleftrightarrow$ *$\bar{R}$ is Prüfer and* $\mathrm{Spec}(\bar{R}) \longrightarrow \mathrm{Spec}(R)$ *is injective.*

4. **Open domains.** By taking $\mathcal{Q}$ = open, we obtain *open domains*; by restricting to all overrings T of R other than its quotient field, we get *propen* (properly open) *domains.* We obtain below a characterization of open domains. As propen domains are treed and semilocal (the latter by virtue of (quasi-) compactness of prime spectra), combinatorial methods work well. If M is a maximal ideal of R we call $\{P \in \mathrm{Spec}(R) : P \subset M\}$ a *branch* of R, and establish

THEOREM 4.1. *R is open* $\Longleftrightarrow$ *R is GD, R is semilocal, and each branch of R is well-ordered under inclusion* $\Longleftrightarrow$ *R is a propen G(oldman)-domain* $\Longleftrightarrow$ $\mathrm{Spec}(V) \longrightarrow \mathrm{Spec}(R)$ *is open for each valuation overring V of R* $\Longleftrightarrow$ $\mathrm{Spec}(T) \longrightarrow \mathrm{Spec}(R)$ *is open for each domain T containing R.*

5. **Local homeomorphism domains.** In this section we consider $\mathcal{Q}$ = local homeomorphism (LH), and study the relationship of LH-domains with previously defined classes of domains. We say R has *finite fibers* if $\mathrm{Spec}(T) \longrightarrow \mathrm{Spec}(R)$ has finite fibers for each overring T of R. We prove

THEOREM 5.1. *R is an LH-domain* $\Longleftrightarrow$ *each overring of R is open* $\Longleftrightarrow$ *R is open, R has finite fibers, and each overring of R is treed.*

6. **Propen not open domains.** In this section we consider domains R which are propen but not open. Using the methods of W. J. Lewis [9], one can construct several pertinent examples of such domains. For a treed domain R, call $\{P \in \mathrm{Spec}(R) : P \subset J(R)\}$ the *trunk* of R [denoted $\mathrm{tr}(R)$]; call the prime $\bigcup\{P \in \mathrm{Spec}(R): P \in \mathrm{tr}(R)\}$ the vertex of R [denoted $v(R)$] and let

$[0, P] = \{Q \in \mathrm{Spec}(R): Q \subset P\}$. We prove

PROPOSITION 6.1. *If R is propen not open, then* $\mathrm{tr}(R)$ *is an infinite set (whence, $v(R) \neq 0$).*

PROPOSITION 6.2. *R is propen not open $\Longleftrightarrow$ is GD, $[0, P]$ is open for each nonzero $P \in \mathrm{Spec}(R)$, and no overring of R other than its quotient field is a G-domain.*

PROPOSITION 6.3. *Let R be local. Then R is propen not open $\Longleftrightarrow$ R is GD, $[0, P]$ is open for each nonzero $P \in \mathrm{Spec}(R)$, and R is not a G-domain.*

We remark that the condition that $[0, P]$ be open, which appears in (6.2) and (6.3), may be characterized without explicit reference to topology [13].

THEOREM 6.4. *R is propen not open and $R/v(R)$ is GD $\Longleftrightarrow$ R is GD, $R_{v(R)}$ is propen not open, and $R/v(R)$ is open.*

The principal applications of Theorem 6.4 are to Bézout domains. By means of Lewis' methods of constructing Bézout domains [9, Theorem 3.1], we infer that the spectrum of an arbitrary propen not open domain is obtained topologically as a quotient space of the disjoint union of the spectrum of an open domain and the spectrum of a local propen not open domain.

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G-TRANSVERSALITY

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Let G be a compact Lie group and N, M and $Y \subset M$ be smooth G manifolds. Suppose $f: N \longrightarrow M$ is a proper G map. We give an obstruction theory (Theorem 1) for a proper G homotopy between f and a map g transverse to Y written $f \pitchfork Y$. In this generality we cannot say more; however, when $f: N \longrightarrow M$ is a quasi-equivalence of G vector bundles over Y, this can be considerably improved (Theorem 2) by removing the dependence of the map f. By definition f is a quasi-equivalence if N and M are G vector bundles over Y and f is proper, fiber preserving and degree 1 on fibers. *To be concise we suppose G is abelian* and omit applications and insights, referring to [1] and [2] for further information.

Let K be a subgroup of G and $\hat{K}$ the set of real irreducible K modules. If Γ and Ω are real K modules, let $V_{\Gamma,\Omega}$ denote the space of surjective real K homomorphisms of Γ to Ω. By Schur's lemma $V_{\Gamma,\Omega} = \Pi_{\psi \in \hat{K}} V^{\psi}_{\Gamma,\Omega}$ where $V^{\psi}_{\Gamma,\Omega}$ has the homotopy type of the Stiefel manifold of b_ψ frames in the D_ψ vector space of dimension a_ψ. Here D_ψ is the division algebra of real K endomorphisms of ψ and $\Gamma = \Sigma_{\psi \in \hat{K}} a_\psi \psi$, $\Omega = \Sigma_{\psi \in \hat{K}} b_\psi \psi$.

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