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ON THE NORM FORM OF A FINITE GALOIS EXTENSION OVER Q

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1. **Introduction.** Let $\lambda: T \longrightarrow T'$ be a **Q**-isogeny of algebraic tori defined over **Q**, the rational number field. Then the isogeny λ induces naturally the following maps (cf. [2]):

$$\lambda_v: T_v \longrightarrow T'_v, \quad \lambda_v^c: T_v^c \longrightarrow T'^c_v, \quad \lambda_{\mathbf{Q}}^\infty: T_{\mathbf{Q}}^\infty \longrightarrow T'^\infty_{\mathbf{Q}}, \quad (\hat{\lambda})_{\mathbf{Q}}: (\hat{T}')_{\mathbf{Q}} \longrightarrow (\hat{T})_{\mathbf{Q}}.$$

For a homomorphism $\alpha: G \longrightarrow G'$ of commutative groups with finite kernel and cokernel, we define the q-symbol of α by $q(\alpha) = [\mathrm{Cok}\ \alpha]/[\mathrm{Ker}\ \alpha]$. Then the q-symbols of the above maps are defined, and $q(\lambda_v^c) = 1$ for almost all finite prime v; more precisely, if K is a finite splitting field for T and T' over **Q**, then $q(\lambda_v^c) = 1$ whenever v is prime to the degree of λ and is unramified relative to $K/\mathbf{Q}$. In [2], we prove

THEOREM 1. *The relative class number $h_T/h_{T'}$ of T, T' over* **Q** *can be expressed as*

$$\frac{h_T}{h_{T'}} = \frac{\tau_T}{\tau_{T'}} \cdot \frac{q(\lambda_\infty)}{q(\lambda_{\mathbf{Q}}^\infty)q((\hat{\lambda})_{\mathbf{Q}})} \cdot \prod_{v \neq \infty} q(\lambda_v^c),$$

where τ_T (resp. $\tau_{T'}$) is the Tamagawa number of T (resp. T') over **Q**.

In this paper, we apply Theorem 1 to the study of the norm form of a finite Galois extension over **Q**.

2. **Main theorem.** Let $K/\mathbf{Q}$ be a Galois extension of finite degree n. Denote by N the norm map $R_{K/\mathbf{Q}}(\mathbf{G}_m) \longrightarrow \mathbf{G}_m$, where $\mathbf{G}_m$ is the multiplicative group of the universal domain Ω, and $R_{K/\mathbf{Q}}$ is the Weil functor of restricting the field of definition from K to **Q** (cf. [3]). We have an exact sequence

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$$\text{(N)} \qquad 0 \longrightarrow \operatorname{Ker} N \xrightarrow{i} R_{K/\mathbf{Q}}(\mathbf{G}_m) \xrightarrow{N} \mathbf{G}_m \longrightarrow 0$$

of tori defined over $\mathbf{Q}$, where i is the canonical inclusion. We attach to (N) a $\mathbf{Q}$-isogeny λ: $R_{K/\mathbf{Q}}(\mathbf{G}_m) \longrightarrow \operatorname{Ker} N \times \mathbf{G}_m$ defined by $\lambda(x) = (x^n N(x)^{-1}, N(x))$. Applying Theorem 1 to the isogeny λ, we obtain

THEOREM 2. *Let K, λ be as above. Then we have*

$$h_K = \frac{h_1}{\tau_1} \cdot \frac{q(\lambda_\infty)}{q(\lambda_{\mathbf{Q}}^\infty) q((\hat{\lambda})_{\mathbf{Q}})} \cdot \prod_{v \neq \infty} q(\lambda_v^c),$$

where h_K is the class number of K, and h_1 (resp. τ_1) is the class number (resp. the Tamagawa number) of the torus Ker N *over* $\mathbf{Q}$.

Let $\{x_1, \ldots, x_n\}$ be an integral basis of K. The form f defined by

$$f(X_1, \ldots, X_n) = N_{K/\mathbf{Q}}(x_1 X_1 + \cdots + x_n X_n)$$

is an integral form in n variables of degree n. The general linear group $GL_n(\Omega)$ acts on the set of forms in n variables as follows: if $u \in GL_n(\Omega)$ and g is a form in n variables, then $(gu)(X_1, \ldots, X_n) = g(Y_1, \ldots, Y_n)$ with $(Y_1, \ldots, Y_n)^t = u(X_1, \ldots, X_n)^t$. We identify the torus $R_{K/\mathbf{Q}}(\mathbf{G}_m)$ with a subgroup of $GL_n(\Omega)$ by means of the basis $\{x_1, \ldots, x_n\}$. Two integral forms g, g' in n variables are said to be in the same K-class if $g' = gz$ with z in the set $R_{K/\mathbf{Q}}(\mathbf{G}_m)_{\mathbf{Z}}$ of elements of $R_{K/\mathbf{Q}}(\mathbf{G}_m) \cap M_n(\mathbf{Z})^\times$. Also, g, g' are said to be in the same K-genus if $g' = gt$ with t in the set $R_{K/\mathbf{Q}}(\mathbf{G}_m)_{\mathbf{Q}}$ of elements of $R_{K/\mathbf{Q}}(\mathbf{G}_m)$ with rational coefficients, and $g' = gu_v$ with $u = (u_v)_v \in R_{K/\mathbf{Q}}(\mathbf{G}_m)_{\mathbf{A}}^\infty = R_{K/\mathbf{Q}}(\mathbf{G}_m)_\infty \times \Pi_{v \neq \infty} R_{K/\mathbf{Q}}(\mathbf{G}_m)_{\mathbf{Z}_v}$, where $R_{K/\mathbf{Q}}(\mathbf{G}_m)_v$ (resp. $R_{K/\mathbf{Q}}(\mathbf{G}_m)_{\mathbf{Z}_v}$) denotes the set of elements of $R_{K/\mathbf{Q}}(\mathbf{G}_m)$ with coefficients in $\mathbf{Q}_v$ (resp. $R_{K/\mathbf{Q}}(\mathbf{G}_m) \cap M_n(\mathbf{Z}_v)^\times$). Let H denote the kernel of the norm map N: $R_{K/\mathbf{Q}}(\mathbf{G}_m) \longrightarrow \mathbf{G}_m$.

MAIN THEOREM. *There exists an injection Ψ of the set of K-classes in the K-genus of f into the quotient space $H_{\mathbf{A}}/H_{\mathbf{A}}^\infty \cdot H_{\mathbf{Q}}$. Moreover, if the class number of K equals* 1, *then Ψ is a bijection and the number of K-classes in the K-genus of f is given by*

$$\text{(1)} \qquad \tau_1 \cdot q(\lambda_{\mathbf{Q}}^\infty) q((\hat{\lambda})_{\mathbf{Q}}) / q(\lambda_\infty) \Pi_p q(\lambda_p^c),$$

τ_1 and the q-symbols being as in Theorem 2.

SKETCH OF THE PROOF. Take a K-class $[g]$ in the K-genus of f. By definition, we have $g = ft$ with $t \in R_{K/\mathbf{Q}}(\mathbf{G}_m)_{\mathbf{Q}}$, and $g = fu_v$ with $u = (u_v)_v$

$\in R_{K/\mathbf{Q}}(\mathbf{G}_m)_{\mathbf{A}}$. This implies that $f = fu_v t^{-1}$ for all v. Putting $s_v = u_v t^{-1}$, we have $s_v \in H_v$ for all v, and $s_v \in H_{\mathbf{Z}_v}$ for almost all finite prime v. Hence, $s = (s_v) \in H_{\mathbf{A}}$. We verify that the map defined by $\Psi([g]) = s(H_{\mathbf{A}}^{\infty} \cdot H_{\mathbf{Q}})$, is the desired injection. Furthermore, suppose that the class number of K is 1, i.e., $R_{K/\mathbf{Q}}(\mathbf{G}_m)_{\mathbf{A}} = R_{K/\mathbf{Q}}(\mathbf{G}_m)_{\mathbf{A}}^{\infty} \cdot R_{K/\mathbf{Q}}(\mathbf{G}_m)_{\mathbf{Q}}$. Take any coset $s(H_{\mathbf{A}}^{\infty} \cdot H_{\mathbf{Q}})$ in $H_{\mathbf{A}}/H_{\mathbf{A}}^{\infty} \cdot H_{\mathbf{Q}}$. Since $s = (s_v)_v \in H_{\mathbf{A}} \subset R_{K/\mathbf{Q}}(\mathbf{G}_m)_{\mathbf{A}}$, we can write $s = ut$ with $u = (u_v) \in R_{K/\mathbf{Q}}(\mathbf{G}_m)_{\mathbf{A}}^{\infty}$ and $t \in R_{K/\mathbf{Q}}(\mathbf{G}_m)_{\mathbf{Q}}$, i.e., $s_v = u_v t$ for all v. Then, $f = fs_v = fu_v t$ because $s = (s_v)_v \in H_{\mathbf{A}}$. From this follows that the K-class of the form g defined by $g = fu_v = ft^{-1}$ is in the K-genus of f, and $\Psi([g]) = s(H_{\mathbf{A}}^{\infty} \cdot H_{\mathbf{Q}})$. The last assertion is an immediate consequence of Theorem 2.

REMARK. If K is a finite abelian extension over Q, the number $\tau_1 \cdot q(\lambda_{\mathbf{Q}}^{\infty})q((\hat{\lambda})_{\mathbf{Q}})/q(\lambda_\infty)\Pi_p q(\lambda_p^c)$ can be effectively computed by means of results in class field theory (cf. [2]). For example, if $K = \mathbf{Q}(\sqrt{m})$ is a quadratic field, we have $\tau_1 = 2$ (cf. [1]), $q((\hat{\lambda})_{\mathbf{Q}}) = 1$, $q(\lambda_\infty) = 1$,

$$q(\lambda_{\mathbf{Q}}^{\infty}) = \begin{cases} 2 & \text{if } m < 0, \text{ or } m > 0 \text{ and } N_{K/\mathbf{Q}}(\epsilon) = -1, \\ 4 & \text{if } m > 0 \text{ and } N_{K/\mathbf{Q}}(\epsilon) = 1, \end{cases}$$

where ϵ is a fundamental unit in K, and $\Pi_p q(\lambda_p^c) = 2^{t+1}$, where t is the number of distinct prime factors of the discriminant d_K of K.

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