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A DOUBLE SCALE OF WEIGHTED L^2 SPACES

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We assemble here for easy reference a number of related results on a certain family of weighted L^2 spaces over R^n. Most of these results are probably familiar to anyone who has thought about them at all, but they do not seem to be readily available in the literature. Proofs are to be submitted to Trans. Amer. Math. Soc.

Let S denote the space of tempered test functions (real or complex-valued) on R^n, and S' the dual space of tempered distributions [3]. Let $\hat{f}$ denote the Fourier transform of the function f. Now define on S the operators X and K by:

$$(Xf)(x) = (1 + x^2)^{1/2} f(x), \quad \widehat{(Kf)}(k) = (1 + k^2)^{1/2} \hat{f}(k)$$

and then define in terms of these operators on S the norms

$$\|f\|_{\alpha,\beta} = \|K^\alpha X^\beta f\|_2, \qquad -\infty < \alpha, \beta < +\infty,$$

$$\|f\|'_{\alpha,\beta} = \|X^\beta K^\alpha f\|_2, \qquad -\infty < \alpha, \beta < +\infty,$$

where $\|f\|_2$ is the usual L^2 norm of f.

Our first result relates these norms:

(1) For each α, β the norms $\| \ \|_{\alpha,\beta}$ and $\| \ \|'_{\alpha,\beta}$ are equivalent.

This follows directly from Leibnitz' rule if α is a positive even integer, but requires some form of interpolation theorem (cf. [2]) for other positive values, and a duality argument for negative values.

Now let $H(\alpha, \beta)$ denote the completion of S in the norm $\| \ \|_{\alpha,\beta}$. It is helpful to think of the family $H(\alpha, \beta)$ as a doubly-indexed scale of weighted L^2 spaces in which, roughly speaking, the second index describes the behavior of the function $f(x)$ as $|x| \longrightarrow \infty$, and the first the behavior of $\hat{f}(k)$ as $|k| \longrightarrow \infty$. It is often useful to consider these two modes of behavior separately, which

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a singly-indexed scale of spaces, such as the L^p spaces, cannot easily do.

The following properties of the spaces $H(\alpha, \beta)$ hold, up to obvious identifications:

(2) For all α, β, $H(\alpha, \beta)$ is a Hilbert space and $S \subset H(\alpha, \beta) \subset S'$.

(3) If $\alpha \geqslant \gamma$, $\beta \geqslant \delta$, then $H(\alpha, \beta) \subset H(\gamma, \delta)$ densely.

(4) If $\alpha > \gamma$, $\beta > \delta$, then $H(\alpha, \beta) \subset H(\gamma, \delta)$ compactly, i.e., sets bounded in $H(\alpha, \beta)$ are compact in $H(\gamma, \delta)$.

(5) $\bigcap\{H(\alpha, \beta): -\infty < \alpha, \beta < +\infty\} = S$. $\bigcup\{H(\alpha, \beta): -\infty < \alpha, \beta < +\infty\} = S'$.

(6) For all α, β, $H(-\alpha, -\beta)$ is the dual of $H(\alpha, \beta)$ under the pairing $(f, g) = \int f(x)g(x)\, dx$.

(7) For all α, β, $H(\beta, \alpha)$ is the Fourier transform of $H(\alpha, \beta)$.

These properties all follow from (1) by elementary calculations.

The following relations with other better-known spaces also hold:

(8) If $1 \leqslant p \leqslant 2$, then $H(0, \beta) \subset L^p \subset H(-\beta, 0)$, where $\beta > n((2-p)/p)/2$. If $2 \leqslant p \leqslant \infty$, then $H(\alpha, 0) \subset L^p \subset H(0, -\alpha)$, where $\alpha > n((p-2)/p)/2$.

(9) If C_k denotes the space of all k-fold continuously differentiable functions on R^n vanishing at ∞, and M_k the dual space of distributions of order k, then $H(k + \alpha, 0) \subset C_k \subset H(k, -\alpha)$, where $\alpha > n/2$; $H(-k, \beta) \subset M_k \subset H(-k-\beta, 0)$, where $\beta > n/2$.

(10) If L^2_k denotes the Sobolev space of all functions $f \in L^2$, whose k-fold derivatives are all square integrable, then $L^2_k = H(k, 0)$.

(11) If L^2_α denotes the Lipschitz space of all functions $f \in L^2$ whose derivatives up to order $k = [\alpha]$ are all square-integrable and satisfy a square-integrable Lipschitz condition of order $\lambda = (\alpha)$ of the form $\|f(\cdot + h) - f(\cdot)\|_2 = O(|h|^\lambda)$ as $|h| \longrightarrow 0$ then $H(\alpha, 0) \subset L^2_\alpha \subset H(\alpha - \epsilon, 0)$, where $\epsilon > 0$.

(12) If R_λ denotes the Riesz space of all functions $f \in L^2$ for which the Riesz potential $(0 < \lambda < n)$

$$\iint \frac{f(x)f(y)}{|x-y|^\lambda}\, dx\, dy < \infty$$

is finite, then $R_\lambda \subset H((\lambda - n)/2, 0)$.

These relations all follow in straightforward fashion from standard estimates of Hölder or Sobolev type.

The spaces $H(\alpha, \beta)$ behave in a natural way under certain standard operations:

(13) If P is the operator defined by multiplication by a polynomial $P(x)$ of degree d, and if D is the associated differential operator, then

$$P: H(\alpha, \beta) \longrightarrow H(\alpha, \beta - d), \quad D: H(\alpha, \beta) \longrightarrow H(\alpha - d, \beta)$$

where it is understood that both operators are *bounded* on the spaces indicated. More generally, the same is true if P is replaced by multiplication by any C^∞ whose behavior as $|x| \longrightarrow \infty$ is dominated by $P(x)$, or if D is replaced by any pseudo-differential operator whose signature is dominated by $P(k)$.

More generally still, we have

(14) If $f \in H(\alpha, \beta)$, $g \in H(\gamma, \delta)$, where $\alpha + \gamma > 0$, then the product fg is well defined, and $fg \in H(\epsilon, \beta + \delta)$, where $\epsilon = \min\{\alpha, \gamma, \alpha + \gamma - n/2\}$.

(15) If $f \in H(\alpha, \beta)$, $g \in H(\gamma, \delta)$, where $\beta + \delta > 0$, then the convolution $f * g$ is well defined, and $f * g \in H(\alpha + \gamma, \epsilon)$, where $\epsilon = \min\{\beta, \delta, \beta + \delta - n/2\}$.

These properties again follow from standard estimates of Hölder or Sobolev type.

Finally, the spaces $H(\alpha, \beta)$ admit a natural interpolation process which has a nice interpretation in the $\alpha - \beta$ plane:

(16) If $\sigma = \lambda\alpha + (1 - \lambda)\gamma$, $\tau = \lambda\beta + (1 - \lambda)\delta$, $0 \leqslant \lambda \leqslant 1$, then $H(\alpha, \beta) \cap H(\gamma, \delta) \subset H(\sigma, \tau)$.

Thus if $f \in H(\alpha, \beta)$ and $H(\gamma, \delta)$, then $f \in H(\sigma, \tau)$ for all (σ, τ) lying on the line segment joining (α, β) and (γ, δ).

Various applications readily present themselves. In particular, the behavior of various singular integral and pseudo-differential operators on the spaces $H(\alpha, \beta)$ can often be determined immediately from the properties listed above, and then reinterpreted in other spaces as required. In particular, the integral equations arising in the quantum theory of scattering admit a specially nice treatment in these spaces, which accounts for the author's original interest. It is clear that other useful variations on this theme may be obtained by replacing the operators X and K by other choices; e.g., if E is a bounded region in R^n with smooth boundary B, then S can be replaced by the space of C^∞ functions defined on E and vanishing to all orders on B, and X by the operation of multiplication by $(1 + \rho^2)^{1/2}$, where $\rho(x)$ is the reciprocal of the distance from x to B. In this setting the $H(\alpha, \beta)$ spaces are again related to the well-known Sobolev spaces W_k^2 as in (10). More generally, if R^n is replaced by any nilpotent Lie group G, X by a suitable function of the distance from the origin, and K by $(1 - \Delta)^{1/2}$, then the spaces $H(\alpha, \beta)$ are related to certain spaces defined by Goodman [1] in his study of distributions on G.

The principal advantages, if any, of the spaces $H(\alpha, \beta)$ over some of these other spaces will stem from the fact that they form a continuous double scale of

Hilbert spaces, compactly nested, which behave nicely under the Fourier transform.

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ADDED IN PROOF. Professor Louis Nirenberg has shown that our first result above can also be obtained directly via standard arguments in the theory of pseudo-differential operators, as in the paper of Kohn and Nirenberg, Comm. Pure Appl. Math. 18 (1965), 269–305, without the need of an interpolation theorem (private communication).

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