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WEAK CONTINUITY OF BANACH ALGEBRA PRODUCTS

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1. This note is based on the remark that a variety of common Banach algebras are dual Banach spaces. For example, a well-known result of Sakai (see [6]) states that von Neumann algebras are characterized, among the C^*-algebras, by this property (see §3 for further examples).

Clearly, in a dual Banach space two additional topologies are ubiquitous companions to the norm: the *weak** and the bounded *weak** topologies (see [1]). Below we announce several theorems about continuity of the product in these topologies. We thereby obtain known as well as (apparently) new results in concrete instances considered by Pym, Rubel, Shields, Ryff, Shapiro, Conway, Dixmier, et al.

Generally we follow [2] for notations and definitions, with the main exception that we will consider a Banach space A whose dual A^* will be denoted by B; a will be a typical element of A and b and c will be typical elements of B. Also, the bounded weak* topology (see [1]) on B will be denoted by ℓw^* (rather than bw^*) in view of the general setting of [3].

2. Let X, Y and Z be linear topological spaces and let f: $X \times Y \to Z$ be a bilinear map. Consider the following properties of f:

[J, (x_0, y_0)]: $f(x, y) \to f(x_0, y_0)$ as $x \to x_0$, $y \to y_0$;

[J]: f satisfies [J, (x_0, y_0)] for all (x_0, y_0);

[L]: $f(x, y_0) \to f(x_0, y_0)$ as $x \to x_0$ for each y_0;

[R]: $f(x_0, y) \to f(x_0, y_0)$ as $y \to y_0$ for each x_0;

then

$$[\mathrm{J}, (0, 0)] + [\mathrm{L}] + [\mathrm{R}] \Longleftrightarrow [\mathrm{J}],$$

and none of the three properties in the left-hand term is redundant in general.

We assume from now on that there is a norm continuous bilinear map

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$B \times A \to A$ which will be denoted by $(a, b) \to b \vdash a$. We dualize $\vdash$ to obtain a bilinear norm continuous map $B \times B \to B$ denoted $(b, c) \longrightarrow bc$ and defined by $\langle a, bc \rangle = \langle b \vdash a, c \rangle$.

Consider for each $a_0 \in A$, the operator $B \to A$ given by $b \to b \vdash a_0$ and the following properties for them:

(w^*-w) w^*-w continuity; (r) the range of the transpose of $b \to b \vdash a_0$ is contained in A; (ℓw^*-n) ℓw^*-norm continuity; (k) compact; (f) finite rank; (wk) weakly compact.

In the next theorem the properties in [ ] refer to the product associated to $\vdash$.

2.1 THEOREM. *For the w^*-topology*

(i) [R] *always holds,*

(ii)
$$\begin{aligned}(f) \Leftrightarrow [J, (0, 0)] \Leftrightarrow [J] \Leftrightarrow (f) + [L] \Leftrightarrow (f) + (w^* - w)\\ \Rightarrow [L] \Leftrightarrow (r) \Leftrightarrow (w^* - w),\end{aligned}$$

and the single implications cannot be reversed in general,

(iii) [L] *holds for each if and only if A is reflexive.*

2.2 THEOREM. *For the ℓw^*-topology*

(i) (k) $\Leftrightarrow$ [J, (0, 0)],

(ii)
$$\begin{aligned}[J] \Leftrightarrow [J, (0, c_0)] \Leftrightarrow [J, (0, 0)] + [L]\\ \Leftrightarrow [L] + (k) \Leftrightarrow (\ell w^* - n),\end{aligned}$$

(iii) [L] $\Longrightarrow$ (wk) *and the implication cannot be reversed in general.*

3. EXAMPLE 1. Let G be a locally compact group, $A = C_0(G)$ the space of continuous functions vanishing at infinity under the sup norm. Then B is the space of finite Radon measures on G. For $f \in A$, $\mu \in B$, take $\mu \vdash f$ to be the function

$$(\mu \vdash f)(s) = \int_G f(ts)\, d\mu(t).$$

The product associated to $\vdash$ is the convolution of finite measures.

EXAMPLE 2. Let Ω be a connected open set of the complex plane supporting nonconstant bounded holomorphic functions (e.g., a disc). Let $B = H^\infty(\Omega)$ be the space of bounded holomorphic functions on Ω under the sup norm. Then (see [5]) $B = H^\infty(\Omega)$ is the dual of $A = L^1(\Omega)/N$ where

$$N = \left\{ f \in L^1(\Omega); \iint fh\,dx\,dy = 0 \text{ for each } h \in H^\infty(\Omega) \right\}.$$

Define $h \vdash (f + N) = hf + N$. The product associated to this $\vdash$ is the ordinary function product. See [5] for a detailed study of this space.

EXAMPLE 3. Let H be a Hilbert space, $B = L(H)$ the space of all bounded linear operators in H. B is the dual of $A = L_*(H)$, the trace class (see [6]). Then, $T \vdash S = ST$ for $T \in L(H)$, $S \in L_*(H)$ induces the operator product in $L(H)$.

EXAMPLE 4. Let $A = \ell_q(S)$, $B = \ell_p(S)$ with $1/p + 1/q = 1$, $p, q > 1$, and $p = \infty$ for $q = 1$. We can consider an arbitrary $\vdash$ product and the pointwise product. The main feature needed here is that all operators $\ell_p \to \ell_q$, $2 < p \leqslant \infty$, are compact (see [4]).

EXAMPLE 5. $A = c_0(S)$, $B = \ell_1(S)$ with pointwise product.

EXAMPLE 6. Let (X, Σ, μ) be a finite measure space, $A = L^1(X, \Sigma, \mu)$ and $B = L^\infty(X, \Sigma, \mu)$. The product here is again the pointwise product.

Then

Ex	[J] for w^*	[J] for ℓw^*
1	holds iff G is finite	holds iff G is compact
2	never holds	always holds
3	holds iff dim $H < \infty$	holds iff dim $H < \infty$
4	holds for pointwise product iff S is finite	holds for all $\vdash$-products if $2 < p \leqslant \infty$; holds for pointwise product if $1 < p \leqslant \infty$
5	holds iff S is finite	always holds
6	holds iff Σ is finite	holds iff μ is purely atomic

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