BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 3, May 1975

VOLTERRA-STIELTJES INTEGRAL EQUATIONS WITH LINEAR CONSTRAINTS AND DISCONTINUOUS SOLUTIONS

BY CHAIM SAMUEL HÖNIG

Communicated by Alberto Calderón, November 11, 1974

X and Y denote Banach spaces; we consider systems of the form

(K) $$y(t) - y(t_0) + \int_{t_0}^{t} d_\sigma K(t, \sigma) \cdot y(\sigma) = f(t) - f(t_0),$$

(F) $$F[y] = c,$$

where $y, f \in G([a, b], X)$ (the space of regulated functions g: $[a, b] \longrightarrow X$, i.e., g has only discontinuities of the first kind); $K \in G^{\text{uo}}$ (see §2) and $F \in L[G([a, b], X), Y]$ (linear constraint). (K) includes linear Volterra integral equations, linear delay differential equations, differential equations $y' + A'y = f^1$, with the meaning that we have

(L) $$y(t) - y(s) + \int_s^t dA(\sigma) \cdot y(\sigma) = f(t) - f(s) \quad \text{for all } s, t \in [a, b].$$

In §2 we give the existence of the resolvent for (K) and in §3 for (L); in §4 we find the Green function for the system (K), (F). The results of §1 are used in the proofs. All results of this announcement may be extended to open intervals and Y a separated sequentially complete locally convex TVS.

The proofs will appear in [H.3].

1. A *division* of $[a, b]$ is a finite sequence d: $t_0 = a < t_1 < \cdots < t_n = b$. We write $|d| = n$ and $\Delta d = \sup_{1 \leq i \leq n} |t_i - t_{i-1}|$. The set D of all divisions of $[a, b]$ is ordered by refinement and $\lim_{d \in D} x_d$ denotes the limit according to the associated net. For α: $[a, b] \longrightarrow L(X, Y)$ and f: $[a, b] \longrightarrow X$ we define the usual Riemann-Stieltjes operator integral

$$\int_a^b d\alpha(t) \cdot f(t) = \lim_{\Delta d \to 0} \sum_{i=1}^{|d|} [\alpha(t_i) - \alpha(t_{i-1})] \cdot f(\xi_i)$$

where $\xi_i \in [t_{i-1}, t_i]$ (see [G], [H.1], [D]), and the *interior integral*

AMS (MOS) subject classifications (1970). Primary 34G05, 45A05, 34A30, 45D05, 34B05; Secondary 26A45, 26A42, 28A45.

$$\int_a^{\cdot b} d\alpha(t) \cdot f(t) = \lim_{d \in D} \sum_{i=1}^{|d|} [\alpha(t_i) - \alpha(t_{i-1})] \cdot f(\xi_i^\bullet)$$

where $\xi_i^\bullet \in]t_{i-1}, t_i[$ (see [K], [H, p. 96]), when these limits exist. The existence of the first integral implies the existence of the second one and reciprocally, if α and f are bounded with no common discontinuity. We define

$$SV[\alpha] = SV_{[a,b]}[\alpha] = \sup_{d \in D} SV_d[\alpha]$$

where

$$SV_d[\alpha] = \sup \left\{ \left\| \sum_{i=1}^{|d|} [\alpha(t_i) - \alpha(t_{i-1})] \cdot x_i \right\| \, | x_i \in X, \|x_i\| \leqslant 1 \right\}.$$

If $SV[\alpha] < \infty$ we say that α is of *bounded semivariation* and we write $\alpha \in SV([a, b], L(X, Y))$; if we have further $\alpha(a) = 0$ we write $\alpha \in SV_0([a, b], L(X, Y))$. For u: $[a, b] \longrightarrow L(X, Y)$ we define $s[u] = \sup_{d \in D} s_d[u]$, where

$$s_d[u] = \sup \left\{ \left\| \sum_{i=0}^{|d|} u(t_i) \cdot x_i \right\| \, | x_i \in X, \|x_i\| \leqslant 1 \right\}$$

and we write $u \in s([a, b], L(X, Y))$ if $s[u] < \infty$. For $f \in G([a, b], X)$ we define $f_-(t) = f(t-)$ if $a \leqslant t \leqslant b$ and $f(a-) = 0$; we write $f \in G_-([a, b], X)$ if $f_- = f$ and $f \in c_0([a, b], X)$ if $f_- = 0$.

THEOREM 1. *The mapping*

$$(\alpha, u) \in SV_0([a, b], L(X, Y)) \times s([a, b], L(X, Y))$$
$$\longmapsto F = F_\alpha + F_u \in L[G([a, b], X), Y]$$

defines a bicontinuous isomorphism of the first Banach space onto the second, where for $f \in G([a, b], X)$ *we define*

$$F_\alpha[f] = \int_a^{\cdot b} d\alpha(t) \cdot f(t) \quad \text{and} \quad F_u[f] = \sum_{a \leqslant t \leqslant b} u(t) \cdot [f(t) - f(t-)].$$

We have $\|F_\alpha\| = SV[\alpha]$, $\alpha(t) \cdot x = F[\chi_{]a,t]}x]$ *and* $u(t) \cdot x = F[\chi_{\{t\}}x]$.

For $X = Y = R$ this theorem is due to Kaltenborn [K].

THEOREM 2. *Given* $\alpha \in SV([c, d], L(X, Y))$, h: $[c, d] \times [a, b] \longrightarrow L(X)$ *which is a regulated function in the first variable and uniformly of bounded semivariation in the second variable (i.e.,* $h^t \in SV([a, b], L(X))$ *for every*

$t \in [c, d]$ and $\sup_{c<t<d} SV[h^t] < \infty$, where $h^t(s) = h(t, s)$) and $g \in G([a, b], X)$ we have $\bar{h} \in SV([a, b], L(X, Y))$, and $\tilde{g} \in G([c, d], X)$, where

$$\bar{h}(s) = \int_c^d d\alpha(t) \cdot h(t, s) \text{ and } \tilde{g}(t) = \int_a^b d_s h(t, s) \cdot g(s),$$

and

$$\int_a^b d_s\left[\int_c^d d\alpha(t) \circ h(t, s)\right] g(s) = \int_c^d d\alpha(t)\left[\int_a^b d_s h(t, s) \cdot g(s)\right] \tag{1}$$

If $[c, d] = [a, b]$ and g is continuous we have the formula of Dirichlet

$$\int_a^b \left[\int_a^s d\alpha(t) \circ h(t, s)\right] dg(s) = \int_a^b d\alpha(t) \cdot \left[\int_t^b h(t, s)\, dg(s)\right]. \tag{2}$$

If $[c, d] = [a, b]$, $\alpha \in A_{\bar{o}}$ (see §3) and $h \in G^{uo}$ (see §2) we have (2).

REMARK. (1) generalizes a theorem of Bray proved for $X = Y = R$ [B].

2. For U: $[a, b] \times [a, b] \longrightarrow L(X)$ we consider the following properties

$$(SV^o) \qquad \lim_{\delta \downarrow 0} SV_{[s-\delta, s+\delta]}[U^t] = 0 \quad \text{for all } s, t \in [a, b],$$

$$(SV^{uo}) \qquad \lim_{\delta \downarrow 0} \sup_{s,t} SV_{[s-\delta, s+\delta]}[U^t] = 0.$$

We write $U \in G^{uo}$ if U is bounded, regulated as a function of the first variable and satisfies (SV^{uo}). G^{uo} is a Banach space when endowed with the norm $|||U||| = \|U\| + \sup_{a<t<b} SV[U^t]$.

THEOREM 3. *Given $K \in G^{uo}$ we have:*

I. *There is one and only one element $R \in G_I^{uo}$ (i.e. $R \in G^{uo}$ and $R(t, t) \equiv I_X$), the resolvent of* (K), *such that*

$$R(t, s) = I_X - \int_s^t d_\sigma K(t, \sigma) \circ R(\sigma, s) \quad \textit{for all } s, t \in [a, b].$$

II. *For every $f \in G([a, b], X)$ the equation* (K) *with $y(t_0) = x$ has one and only one solution $y \in G([a, b], X)$ given by*

$$y(t) = R(t, t_0)x + \int_{t_0}^t R(t, \sigma)\, df(\sigma)$$

and y depends continuously on f, x and K.

III. *If $K \in G_0^{uo}$ (i.e. $K \in G^{uo}$ and $K(t, t) \equiv 0$) we have*

$$R(t, s) = I_X + \int_s^t R(t, \sigma) \circ d_\sigma K(\sigma, s) \quad \textit{for all } s, t \in [a, b].$$

IV. *The mapping* $K \in G_0^{uo} \mapsto R \in G_I^{uo}$ *is a bicontinuous (nonlinear) bijection from the first space onto the second.*

REMARK. Theorem 3 remains true if we replace G^{uo} by its subspace E^{uo} of continuous functions, by its subspace E^{co} of functions U that satisfy

$$\text{(SV}^c\text{)} \qquad \lim_{t \to t_1} SV[U^t - U^{t_1}] = 0 \quad \text{for every } t_1 \in [a, b],$$

by the corresponding spaces of functions of bounded variation, etc.

3. We now particularize Theorem 3 to (L). We fix a point $\bar{\sigma} \in [a, b]$; given A: $[a, b] \longrightarrow L(X)$ we write $A \in A_{\bar{\sigma}}$ if $A(\bar{\sigma}) = 0$ and if A satisfies (SV^o). (SV$_{uo}$), (SV$_o$), (SV$_c$) denote the analogous for the first variable of the properties (SV^{uo}), (SV^o), (SV^c) in the second variable. We say that R: $[a, b] \times [a, b] \longrightarrow L(X)$ is *harmonic*, and we write $R \in H$, if R satisfies (SV^{uo}), (SV^c), (SV$_{uo}$), (SV$_c$) and

$$\text{(o)} \quad R(t, t) \equiv I_X, \qquad R(t, \sigma) \circ R(\sigma, s) = R(t, s) \quad \text{for all } s, \sigma, t \in [a, b].$$

H^{co} denotes H with the topology induced by E^{uo}; analogously we define H_{co}. The next theorem extends Theorems 3.2 and 3.3 of [M].

THEOREM 4. A. *Given* $A \in A_{\bar{\sigma}}$ *we have:*

I. *There is one and only one* $R \in H$, *the resolvent of* A, *such that*

$$R(t, s) = R(\tau, s) - \int_\tau^t dA(\tau) \circ R(\tau, s) \quad \textit{for all } s, \tau, t \in [a, b].$$

II. *For every* $f \in G([a, b], X)$ *the equation* (L) *with* $y(s) = x$ *has one and only one solution* $y \in G([a, b], X)$ *given by*

$$y(t) = R(t, s)x + \int_s^t R(t, \sigma)\, df(\sigma)$$

and y *depends continuously on* f, x *and* A.

III. $A(t) = \int_t^{\bar{\sigma}} d_\sigma R(\sigma, s) \circ R(s, \sigma)$ *for all* $s \in [a, b]$ *and*

$$R(t, s) = R(t, \sigma) + \int_\sigma^s R(t, \tau) \circ dA(\tau) \quad \textit{for all } s, \sigma, t \in [a, b].$$

B. *If* R: $[a, b] \times [a, b] \longrightarrow L(X)$ *satisfies* (o) *and* (SV_o) *then* $R \in H$ *and* R *is the resolvent of* A *given in* III.

C. *On* H *the topologies of* H^{co} *and* H_{co} *coincide and the mapping* $A \in A_{\bar{\sigma}} \mapsto R \in H$ *is a bicontinuous (nonlinear) bijection from the first space onto the second.*

4. We now consider the problem (K), (F) with $K \in \mathcal{E}^{uo}$; we write $K[y] = f$ for (K) and define $Y_0 = F[K^{-1}(0)]$. Let α be associated to F by Theorem 1; for $s \in [a, b]$ we define $J(s) = \int_a^b d\alpha(t) \circ R(t, s)$.

THEOREM 5. *The following properties are equivalent*:

(i) $y \equiv 0$ *is the only solution of the system* $K[y] \equiv 0, F[y] = 0$.

(ii) $J(t_0)\colon X \longrightarrow Y_0$ *is a continuous bijection.*

From now on we suppose that the equivalent properties (i), (ii) *are satisfied and that*

$$\left\{\int_a^b d\alpha(t) \cdot f(t) \mid f \in G([a, b], X)\right\} = Y_0.$$

We define

$$\bar{J}(t) = R(t, t_0) \circ J(t_0)^{-1}\colon Y_0 \longrightarrow X$$

and

$$G(t, s) = \bar{J}(t) \circ \int_a^s d\alpha(\tau) \circ R(\tau, s) - Y(s - t_0)\bar{J}(t) \circ J(s)$$
$$+ [Y(s - t_0) - Y(s - t)]\, R(t, s).$$

THEOREM 6. A. *The system* $K[y] = g$, $F[y] = c$ *has a solution* $y \in C([a, b], X)$ *iff* $(g, c) \in C([a, b], X) \times Y_0$; *then this solution is*

$$y(t) = \bar{J}(t)c + \int_a^b G(t, s)\, dg(s).$$

B. *The system* $K[y] = f$, $F[y] = c$ *has a solution* $y \in G([a, b], X)$ *iff* $c - F(f) \in Y_0$; *then this solution is given by*

$$y(t) = f(t) + \bar{J}(t)[c - F(f)] - \int_a^b G(t, s) d_s\left[\int_{t_0}^s d_\sigma K(s, \sigma) \cdot f(\sigma)\right].$$

THEOREM 7. *The Green function* $G\colon [a, b] \times [a, b] \longrightarrow L(X)$ *has the following properties*:

(G_0) $F[G_s] = 0$ *for every* $s \in [a, b]$, *where* $G_s(t) = G(t, s)$.

(G_1) $G_s(t) - G_s(t_0) + \int_{t_0}^t d_\sigma K(t, \sigma) \circ G_s(\sigma) = [-Y(s-t) + Y(s-t_0)]I_X$.

(G_2) $\tilde{G}^t(s) + \int_a^s \tilde{G}^t(\sigma) \circ d_\sigma K(\sigma, s) = \bar{J}(t) \circ \alpha(s)$ *where*

$$\tilde{G}(t, \sigma) = G(t, \sigma) + Y(\sigma - t)R(t, \sigma) + Y(\sigma - t_0)[\bar{J}(t) \circ J(\sigma) - R(t, \sigma)].$$

(G_3) *For every* $s \in [a, b]$, G_s *is continuous for* $t \neq s$.

(G_4) G *is uniformly of bounded semivariation in the second variable*; $G(t, b) \equiv 0$; $G(t, a) = 0$ *for* $a < t \leqslant b$, $G(a, a) = -I_X$.

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INSTITUTO DE MATEMÁTICA E ESTATISTICA, UNIVERSIDADE DE SÃO PAULO, SÃO PAULO, BRAZIL