BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 3, May 1975

EXTENSIONS OF THE HASSE NORM THEOREM

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Communicated by Barbara Osofsky, February 4, 1975

1. **Introduction.** In two works, one in 1930 [5, p. 38] and the other in 1931 [6], H. Hasse produced one of the major theorems of class field theory, namely, his norm theorem which states that if k is a cyclic extension of the number field k' then an element of k' is the norm of an element of k if and only if it is the norm of an element everywhere locally. Also in 1931 [6, p. 68], Hasse showed that for the fields $k' = Q$ (the rationals) and $k = Q(\sqrt{13}, \sqrt{-3})$, his norm theorem did not hold, and hence his theorem, unlike all other major results of class field theory, is not true for arbitrary abelian extensions. In the 1967 publications [1, p. 360] from the Brighton Conferference, J. Tate and J.–P. Serre presented $k = Q(\sqrt{13}, \sqrt{17})$ as another example where the Hasse norm theorem does not hold. In 1971 Y. Furuta produced an equation [4, p. 321] which, were it not for an annoying factor in the denominator, would show when the Hasse norm theorem held for k/Q in terms of the central class number and the genus number of k. See also a very interesting result of O. Taussky-Todd [7, Theorem 5]. It is only natural to ask the following question. For which noncyclic extensions of the rationals Q does the Hasse norm theorem hold? The aim of this note is to present theorems which give a computable answer to this question for a certain class of noncyclic extensions of Q. Detailed proofs of the theorems will appear elsewhere.

2. **A new characterization of the Hasse norm theorem.** *Let k be a finite abelian extension of Q.*

Let p be a prime divisor in Q. Let $\beta \in Q^* = Q\backslash\{0\}$. Let $\left(\frac{\beta, k}{p}\right)$ be the Hasse norm residue symbol. Let N be the norm map from k to Q. By abuse of language we will take the statement "the Hasse norm theorem holds for k" to mean that for each $\beta \in Q^*$ there exists $\hat{\beta} \in k$ such that $N\hat{\beta} = \beta$ if and only if $\left(\frac{\beta, k}{p}\right) = 1$ for all prime divisors p of Q.

Let K' be the "narrow" genus field of k, i.e. the maximal abelian ex-

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AMS (MOS) subject classifications (1970). Primary 12A65; Secondary 12A50.

tension of k which is unramified at all (finite) prime ideals of k and which is abelian over Q. Let $\bar{K}$ be the "narrow" central class field of k, i.e. the maximal abelian extension of k which is unramified at all (finite) prime ideals of k and which is a Galois extension of Q such that $G(\bar{K}/k)$ is a subset of the center of $G(\bar{K}/Q)$. Let $g_k^+ = [K'\colon k]$, the "narrow" genus number, and let $z_k^+ = [\bar{K}\colon k]$, the "narrow" central class number. Using these numbers instead of the genus number and the central class number, the annoying factor in the denominator of Furuta's equation can be made to disappear and the following criterion can be proved which states (in a theoretical sense) when the Hasse norm theorem holds.

THEOREM 1. *The Hasse norm theorem holds for k if and only if* $z_k^+ = g_k^+$.

REMARK. From this one can easily prove H. Hasse's original theorem for k/Q, i.e. if k/Q is cyclic then the Hasse norm theorem holds for k.

3. **Computable criteria for the Hasse norm theorem to hold.** We now take a specific type of field for which we can change the criterion of Theorem 1 into computable criteria. Let k be an abelian extension of Q such that $[k\colon Q]$ is a power of some prime l and k is the composite of fields with (plus or minus) odd prime power discriminants, e.g. $k = Q(\sqrt{5}, \sqrt{13})$. All theorems which follow (except Theorem 5) refer only to this type of field. For this type of field one can use the ideas and results contained in A. Fröhlich's two brilliant papers [2], [3] to prove the following theorems.

If precisely one prime divides the discriminant of k then the Hasse norm theorem holds by the remark after Theorem 1.

THEOREM 2. *Suppose precisely two distinct odd primes p_1 and p_2 divide the discriminant of k. Then the Hasse norm theorem holds if and only if* (i) *in the case $p_1 \neq l$ and $p_2 \neq l$ we have either $x^l \equiv p_1 \bmod p_2$ or $x^l \equiv p_2 \bmod p_1$ does not have a solution x;* (ii) *in the case $p_1 = l$ we have either $x^l \equiv p_1 \bmod p_2$ or $x^l \equiv p_2 \bmod p_1^2$ does not have a solution x.*

COROLLARY. *Let $k = Q(\sqrt{p}, \sqrt{q})$ where p and q are primes both congruent to* 1 mod 4. *Then the Hasse norm theorem holds if and only if* $\left(\frac{p}{q}\right) = -1$ *where* $\left(\frac{\cdot}{\cdot}\right)$ *is the Legendre symbol.*

EXAMPLES. By Theorem 2 and the Corollary, the Hasse norm theorem does not hold for $k = Q(\sqrt{13}, \sqrt{-3})$ (H. Hasse's original example) and for $k = Q(\sqrt{13}, \sqrt{17})$ (J. Tate and J.-P. Serre's example). But the Hasse norm theorem does hold for $k = Q(\sqrt{5}, \sqrt{-3})$ and $k = Q(\sqrt{5}, \sqrt{13})$.

Suppose precisely three distinct odd primes p_1, p_2 and p_3 divide the discriminant of k. If $p_i \neq l$ (resp. $p_i = l$) let α_i be an integer whose multiplicative order mod p_i (resp. mod p_i^2) is $p_i - 1$ (resp. $p_i(p_i - 1) = l(l-1)$). Let a_{ij} for $1 \leqslant i, j \leqslant 3$ be defined as follows

$$p_j \equiv \begin{cases} \alpha_i^{a_{ij}} \bmod p_i & \text{if } p_i \neq l, \\ \alpha_i^{a_{ij}} \bmod p_i^2 & \text{if } p_i = l. \end{cases}$$

Let $\bar{a}_{ij}$ be a_{ij} mod l. Let D be the following 3×3 matrix over F_l, the Galois field with l elements.

$$D = \begin{bmatrix} -\bar{a}_{21} & -\bar{a}_{31} & 0 \\ \bar{a}_{12} & 0 & -\bar{a}_{32} \\ 0 & \bar{a}_{13} & \bar{a}_{23} \end{bmatrix}.$$

THEOREM 3. *Suppose precisely three odd primes p_1, p_2, and p_3 divide the discriminant of k. Then the Hasse norm theorem holds if and only if the determinant of D is not* 0.

COROLLARY. *Let $k = Q(\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3})$ where each p_i* $(1 \leqslant i \leqslant 3)$ *is a prime congruent to* 1 mod 4. *Then the Hasse norm theorem does not hold for k.*

EXAMPLE. If m is a positive integer let $Q_m = Q(e^{2\pi i/m})$. Let k_1 be the field in Q_7 such that $[k_1 \colon Q] = 3$. Let k_2 be the field in Q_{13} such that $[k_2 \colon Q] = 3$. Let k_3 be the field in Q_{37} such that $[k_3 \colon Q] = 9$. Let $k = k_1 \cdot k_2 \cdot k_3$. Then one can use Theorem 3 to show the Hasse norm theorem holds for k.

THEOREM 4. *Suppose four or more distinct primes divide the discriminant of k. Then the Hasse norm theorem does not hold for k.*

Theorems 2–4 give a complete answer to when the Hasse norm theorem holds for fields of the type described at the beginning of this section.

Now it is natural to ask if we can get criteria for fields other than those described at the beginning of this section. Along these lines the following theorem will produce a train of obvious corollaries to Theorems 2–4.

THEOREM 5. *Let k and k' be arbitrary finite abelian extensions of Q and suppose $k \supset k'$. If the Hasse norm theorem holds for k then it holds for k'.*

REFERENCES

1. J. W. S. Cassels and A. Fröhlich, *Algebraic number theory*, Thompson Book, Washington, D. C., 1967.

2. A. Fröhlich, *On fields of class two,* Proc. London Math. Soc. (3) 4 (1954), 235–256. MR 16, 116.

3. ———, *On the absolute class-group of abelian fields,* J. London Math. Soc. 29 (1954), 211–217. MR 16, 573; 19, 1431.

4. Y. Furuta, *Über die Zentrale Klassenzahl eines relativ-galoisschen Zahlkörpers,* J. Number Theory 3 (1971), 318–322. MR 45 #6795.

5. H. Hasse, *Bericht über neuere Untersuchungen und Probleme aus der Theorie der algebraischen Zahlkörper.* II, Jber. Deutsch. Math. Verein. 1930, 3 Auflage.

6. ———, *Beweis eines Satzes und Widerlegung einer Vermutung über das allgemeine Normenrestsymbol,* Nachr. Ges. Wiss. Göttingen Math.-Phys. Kl. H. 1 (1931), 64–69.

7. O. Taussky-Todd, *Additive commutators of rational* 2×2 *matrices* (to appear).

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