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THE CAUCHY PROBLEM FOR A FIRST ORDER SYSTEM OF ABSTRACT OPERATOR EQUATIONS

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1. **Introduction.** Recently, Carroll [1], [2], and Hersh [5] have studied questions of representation, existence and uniqueness of solutions for the operator differential equation $Q(d/dt, A)u(t) = 0$, where $Q(r, s)$ is a polynomial in r and s and A is a group generator. Their uniqueness results for the Cauchy problem for this equation involve the additional assumption that A^* also generates a group. Earlier results of Hille [6] for the equation

$$[(d^2/dt^2) - A^2]\, u(t) = 0 \tag{Δ}$$

suggests that this assumption is not necessary for uniqueness. For this equation he shows that a necessary and sufficient condition for the Cauchy problem for (Δ) to have a unique solution is that A generates a group.

In this research announcement we describe a new method for studying questions of uniqueness, representation, and existence for a Cauchy problem for a first order system of differential equations with operator coefficients. This method which employs Schwartz vector valued distributions and the Fourier transformation provides rather complete results for this class of first order systems.

2. **The first order system.** Let $P(s, t)$ be an $n \times n$ matrix whose elements are polynomials in s with coefficients depending continuously upon the parameter t; let A be the generator of a strongly continuous group $G_A(x)$, $-\infty < x < \infty$, of bounded linear operators on the Banach space Y; and let $\vec{u}(t)$ be an n-vector whose components are Y-valued functions of t for $t \geqslant 0$. We con-

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sider the following Cauchy problem for the matrix P:

$$\text{(I)} \qquad \vec{u}\,'(t) = P(-iA, t)\vec{u}(t), \qquad \vec{u}(0) = \vec{\phi}$$

where $\vec{\phi}$ is an n-vector whose components are elements of $\mathcal{D}^\infty(A)$.

In conjunction with (I) we consider the initial value problem, which in matrix notation has the form

$$\text{(II)} \qquad \begin{aligned} (d/dt)R(s, a, t) &= P(s, t)R(s, a, t) \\ R(s, a, a) &= E, \text{ the identity matrix.} \end{aligned}$$

Let α denote the highest order of all the entire functions of s appearing as elements of R, the solution matrix of (II). This number will be called the order of (I). For each $t \geqslant 0$, let $\lambda_j(s, t)$ denote the eigenvalues of the matrix $Q(s, t)$ defined implicitly by $R(s, 0, t) = e^{Q(s,t)}$, and define Λ by $\Lambda(s, t) = \max[\operatorname{Re} \lambda_j(s, t)\colon j = 1, 2, \ldots, n]$.

Finally, it is convenient to introduce here the space of test functions Z_p^p $(p > 1)$ which consists of all entire functions $f(z)$ such that $|f(z)| \leqslant K \exp[\epsilon C|z|^p]$ where C is a positive constant and $\epsilon = 1$, for z nonreal and $\epsilon = -1$ for z real. Convergence is defined as being uniform on compact sets and with uniform maintenance of a bound. The space of all continuous linear operators from Z_p^p to Y is denoted by $T(Z_p^p)$ and the space of all n-vectors whose components are elements of $T(Z_p^p)$ is denoted by $T^{(n)}(Z_p^p)$.

3. Basic theorems.

THEOREM 1 (EXISTENCE AND REPRESENTATION). *Let* $0 < \sigma < 1/\alpha$ *where* α *is the order of the system* (I). *Furthermore, for each* $t \geqslant 0$, *let* $\Lambda(s, t)$ *satisfy for real* $s = \tau$ *the inequality* $\Lambda(\tau, t) < -C|\tau|^\alpha + C_1$ *with* $C > 0$.

(a) *Then the system* (I) *has a solution* $\vec{u}(t)$ *such that* $G_A(x)\vec{u}(t)$ *is in* $T^{(n)}(Z_{1+\sigma}^{1+\sigma})$ *for each* $t \geqslant 0$.

(b) *A representation for this solution is provided by*

$$\vec{u}(t) = \int_{-\infty}^{\infty} \widetilde{R}(\xi, t)G_A(-\xi)\vec{\phi}\,d\xi$$

where $\widetilde{R}(\xi, t)$ *is the inverse Fourier transformation of* $R(s, 0, t)$ *and is to be interpreted in the sense of the theory of distributions.*

THEOREM 2 (UNIQUENESS). *If the elements of the matrix* $R(s, 0, t)$ *are multipliers for* $T^{(n)}(Z_{1+1/\sigma}^{1+1/\sigma})$ *for any* t *with* $0 \leqslant t \leqslant a$, *then the system* (I) *can have at most one solution* $\vec{u}(t)$ *such that* $G_A(x)\vec{u}(t)$ *is in* $T^{(n)}(Z_{1+\sigma}^{1+\sigma})$.

Upon employing Theorem 2, one can now formulate a uniqueness criterion for strict solutions of the Cauchy problem (I).

COROLLARY. *The Cauchy problem* (I) *can have at most one strict solution.*

4. **Sketch of the proofs.** The proofs of these theorems are based upon an inverse separation of variables method which transforms (I) into a Cauchy problem for a first order system of partial differential equations in a Banach space Y. We regard the resulting system of partial differential equations as a system of equations in $T^{(n)}(Z_p^p)$, the unknown function being replaced by an unknown vector valued distribution. Then one employs the Fourier transformation and then follows the usual procedure. Details of these results, together with applications and extensions will appear elsewhere [3].

5. **Remarks.** The Corollary extends the sufficiency part of Hille's result for (Δ) to first order systems. For the case where A is the differential operator $\partial/\partial x$, the system in (I) was investigated already by Gel′fand and Šilov [4]. Therefore, our results may be viewed as a generalization of the classical results of these two authors.

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