8. ———, *Contract law and the value of a game*, Israel J. Math. **5** (1967), 135–144. MR **39** #7947.

9. W. F. Lucas, *The proof that a game may not have a solution*, Trans. Amer. Math Soc. **137** (1969), 219–229. MR **38** #5474.

10. J. von Neumann and O. Morgenstern, *Theory of games and economic behavior*, Princeton Univ. Press, Princeton, N.J., 1944. MR **6**, 235.

11. R. Selten, *Valuation of n-person games*, Advances in Game Theory, Princeton, N.J., 1964, pp. 577–626. MR **29** #1081.

12. L. S. Shapley, *A value for n-person games*, Contributions to the Theory of Games, vol. 2, Princeton Univ. Press, Princeton, N.J., 1953, pp. 307–317. MR **14**, 779.

13. ———, *A solution containing an arbitrary closed component*, Contributions to the Theory of Games, vol. 4, Princeton Univ. Press, Princeton, N.J., 1959, pp. 87–94. MR **22** #635.

14. ———, *Values of large market games: Status of the problem*, RM-3957-PR, The Rand Corporation, Santa Monica, 1964.

Convolution equations and projection methods for their solution, by I. C. Gohberg and I. A. Fel′dman, American Mathematical Society Translations, vol. 41, 1974, ix+261 pp.

Suppose $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are Banach spaces, $\{P_\tau\}$ and $\{Q_\tau\}$ are families of projection operators on $\mathfrak{B}_1$ and $\mathfrak{B}_2$ respectively which converge strongly as $\tau\to\infty$ to the respective identity operators, and A is a bounded linear transformation from $\mathfrak{B}_1$ to $\mathfrak{B}_2$. One says that the projection method (P_τ, Q_τ) is applicable to A if, roughly speaking, $(Q_\tau AP_\tau)^{-1}$ converges strongly to A^{-1} as $\tau\to\infty$. More precisely what is required is that $Q_\tau AP_\tau$, as an operator from $P_\tau\mathfrak{B}_1$ to $Q_\tau\mathfrak{B}_2$, be invertible for sufficiently large τ and that $(Q_\tau AP_\tau)^{-1}Q_\tau$ converge strongly as $\tau\to\infty$. (Then A is necessarily invertible and the strong limit is A^{-1}.)

To give an example, the prototype of those considered in this book, let a be a bounded function defined on the unit circle having Fourier coefficients a_k ($k=0, \pm 1, \cdots$), and consider the operator A on l_2 of the positive integers defined by

$$A\{\xi_j\} = \left\{\sum_{k=1}^{\infty} a_{j-k}\xi_k\right\}_{j=1}^{\infty}.$$

This is the (semi-infinite) Toeplitz operator associated with a. The projections are the simplest ones: $P_n=Q_n=$projection on the subspace of sequences $\{\xi_j\}$ satisfying $\xi_j=0$ for $j>n$. The operator P_nAP_n may then be represented by the finite Toeplitz matrix

$$A_n = (a_{j-k})_{j,k=1}^{n}$$

and the question is whether these matrices are invertible from some n onward and, if so, whether the inverses of these matrices converge strongly as $n\to\infty$ to A^{-1}. It is a surprising fact that if a is continuous then the invertibility of A is a sufficient as well as necessary condition for the applicability of the projection method (P_n, P_n) to A.

Of course this is what one hopes is true. To see why it is surprising take the case of n odd, $n=2m+1$. Then there is no difference between A_n and the matrix $(a_{j-k})_{j,k=-m}^{m}$ whose natural limit is the doubly infinite Toeplitz operator defined on l_2 of the integers by

$$A\{\xi_j\} = \left\{\sum_{k=-\infty}^{\infty} a_{j-k}\xi_k\right\}_{j=-\infty}^{\infty}.$$

Even if this operator is invertible the obvious projection method (corresponding to the subspaces of sequences $\{\xi_j\}$ satisfying $\xi_j=0$ for $|j|>m$) will probably fail. The invertibility of an operator is by no means sufficient for a given projection method to be applicable to it.

Although projection methods clearly have application to numerical computation of A^{-1}, the main interest of the results contained in this book lies in the information they give about the truncated operator $P_\tau A Q_\tau$ which could be very difficult to handle for large finite τ. This is certainly the case for Toeplitz operators. For the most part the operators considered in the book are generalizations, of various kinds, of Toeplitz operators.

Here is an outline of the various chapters.

I. **General theorems concerning Wiener-Hopf equations.** These are the continuous analogues of Toeplitz operator equations, although the two terms are often used interchangeably. This chapter is in fact concerned with a generalization of both, which is obtained as follows.

Let V be a left-invertible operator on a Banach space $\mathfrak{B}$ such that both V and a left inverse $V^{(-1)}$ have spectrum contained in the closed unit disc. For any polynomial $p(\zeta)$ in ζ and ζ^{-1} the operator $p(V)$ is defined in the obvious way and it turns out that its spectral radius is exactly $\max_{|\zeta|=1}|p(\zeta)|$. This fact is used to associate a unique continuous function with each operator in the closure (in the operator norm) of the set of all $p(V)$. This continuous function a is called the *symbol* of the operator and the operator itself is denoted by $a(V)$.

To obtain the Toeplitz operators let $\mathfrak{B}$ be l_2 of the positive integers and V and $V^{(-1)}$ the shifts

$$V\{\xi_1, \xi_2, \cdots\} = \{0, \xi_1, \xi_2, \cdots\}, \qquad V^{(-1)}\{\xi_1, \xi_2, \cdots\} = \{\xi_2, \xi_3, \cdots\}.$$

The set of symbols obtained by the process described above consists of all continuous functions a on the unit circle, and $a(V)$ is just the Toeplitz operator associated with a. (However if $\mathfrak{B}$ is l_1 of the positive integers the symbols are exactly the functions with absolutely convergent Fourier series. The symbols always form a subset, usually proper, of the set of all continuous functions.)

It is not as obvious how one obtains Wiener-Hopf operators. In fact V and $V^{(-1)}$ are defined by

$$\begin{aligned}(V\varphi)(t) &= \varphi(t) - 2\int_0^t e^{s-t}\varphi(s)\,ds,\\ (V^{(-1)}\varphi)(t) &= \varphi(t) - 2\int_t^\infty e^{t-s}\varphi(s)\,ds,\end{aligned} \qquad 0 \leqq t < \infty.$$

If $V^{(n)}$ means V^n for $n \geqq 0$ and $(V^{(-1)})^{-n}$ for $n<0$ then

$$(V^{(n)}\varphi)(t) = \varphi(t) - \int_0^\infty l_n(t-s)\varphi(s)\,ds$$

where

$$1 - \int_{-\infty}^\infty l_n(t)e^{i\lambda t}\,dt = \left(\frac{\lambda - i}{\lambda + i}\right)^n.$$

From this it follows that the Wiener-Hopf operator

$$\varphi \to c\varphi(t) - \int_0^\infty k(t-s)\varphi(s)\,ds, \qquad 0 \leqq t < \infty,$$

has symbol $R(\zeta)$, where

$$c - \int_{-\infty}^\infty k(t)e^{i\lambda t}\,dt = R\left(\frac{\lambda - i}{\lambda + i}\right).$$

Still another example is provided by the shift operator V on $L_2(0, \infty)$:

$$\begin{aligned}(Vf)(t) &= f(t-1), & t > 1,\\ &= 0, & t < 1,\end{aligned}$$

with left inverse $(V^{(-1)}f)(t) = f(t+1)$. The symbols are again all continuous functions, and if the symbol a has Fourier coefficients a_k the corresponding operator $A = a(V)$ is given by

$$(Af)(t) = \sum_{k=-\infty}^{\infty} a_k f(t-k).$$

The main result of the first part of the chapter is that in the general context described above (but with certain other assumptions on V and $V^{(-1)}$) the operator $a(V)$ is invertible on at least one side if and only if $a(\zeta) \neq 0$ and that if this condition is satisfied then $A = a(V)$ is invertible, invertible only on the left, or invertible only on the right depending on whether the number $(2\pi)^{-1}\,\Delta \arg a(\zeta)$ is zero, positive, or negative. (This is rephrased as follows: *The invertibility of A is determined by its symbol.*) This generalizes the familiar index theorem for Toeplitz operators.

The second part of the chapter concerns the explicit inversion of these operators. This is intimately connected with the existence of a factorization of the symbol $a(\zeta)$ in the form $a(\zeta)=a_-(\zeta)\zeta^\kappa a_+(\zeta)$ where $a_-(\zeta)$ and $a_+(\zeta)$ continue to be analytic and nonzero in the regions $|\zeta|>1$ and $|\zeta|<1$ respectively and κ is an integer. This is the well-known Wiener-Hopf factorization. Such a factorization is shown always to exist, if $a(\zeta)\neq 0$, in the context of what are called *decomposing R-algebras*. These are Banach algebras of continuous functions on the unit circle in which the rational functions with poles off the unit circle form a dense subalgebra and for which the conjugate operator (in the sense of Fourier series) is bounded. The Wiener algebra of functions with absolutely convergent Fourier series is the most important example.

II. **Galerkin's method and projection methods of solving linear equations.** Galerkin's method differs from the projection method in that the use of projections is replaced by the use of subspaces. The Galerkin method seems broader (since projections correspond to complemented subspaces) but it is quickly shown that if a given Galerkin method is applicable to a single invertible linear transformation then the subspaces are necessarily complemented.

This chapter sets out the main general results of the theory of projection methods of which we mention only the basic stability theorem: The class of linear transformations for which a given projection method (P_τ, Q_τ) is applicable is closed under both small perturbations and compact perturbations which preserve invertibility.

III. **Projection methods of solving the Wiener-Hopf equation and its discrete analogue.** The general results of the preceding chapter are used to prove the applicability of the natural projection methods for Toeplitz and Wiener-Hopf operators. One consequence is the Szegö limit theorem

$$\lim_{n\to\infty}\frac{D_n(a)}{D_{n-1}(a)} = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\log a(e^{i\varphi})\,d\varphi\right\}$$

for the Toeplitz determinants

$$D_n(a) = |a_{j-k}|_{j,k=1}^{n}$$

proved under the assumptions that a is continuous and $a(\zeta)\neq 0$, $\Delta \arg a(\zeta)=0$.

The second part of the chapter is devoted to the explicit inversion of finite Toeplitz matrices and their continuous analogues. This constitutes the last three sections of the chapter and is a rewriting of the last two sections in the original Russian edition of the book.

IV. **Wiener-Hopf equations with discontinuous functions.** The general theory of Chapter I does not directly apply here. However if V is an isometric operator on Hilbert space there is a reasonable way to define $a(V)$ for any bounded function a defined on the unit circle: apply a to a unitary dilation of V and then compress. If a is piecewise continuous there is a natural way to define the symbol of $a(V)$. (The curve corresponding to the symbol is obtained from the curve corresponding to a by adding line segments between the right and left hand limits of a at its discontinuities.) In this context it remains true that the invertibility of $a(V)$ is determined by its symbol. The usual applications are given to the special cases of Toeplitz and Wiener-Hopf operators, and to the applicability of the projection methods.

V. **Pair equations.** A pair operator is one of the form $PA+QB$ or $AP+BQ$ where A and B are operators on a Banach space, P is a projection, and $Q=I-P$. A pair equation in the context of Toeplitz operators is given by the system

$$\sum_{k=0}^{\infty} a_{j-k}\xi_k + \sum_{k=-\infty}^{-1} b_{j-k}\xi_k = \eta_j \qquad (-\infty < j < \infty).$$

The symbol is now a pair of continuous functions $(a(\zeta), b(\zeta))$. The operator is invertible on at least one side if and only if $a(\zeta)\neq 0$ and $b(\zeta)\neq 0$ in which case the operator is invertible, invertible only on the left, or invertible only on the right depending on whether $(2\pi)^{-1}\,\Delta \arg a(\zeta)/b(\zeta)$ is zero, positive, or negative.

This chapter develops a functional calculus for pair operators analogous to the theory of Chapter I and general results are obtained of which the one quoted is a special case.

VI. **Projection methods for solving pair equations.** Results for pair equations are obtained analogous to those of Chapter III.

VII. **Wiener-Hopf integral-difference equations.** This chapter treats equations of the form

$$\sum_{j=-\infty}^{\infty} a_j\varphi(t-\delta_j) + \int_0^{\infty} k(t-s)\varphi(s)\,ds = f(t) \qquad (0 < t < \infty)$$

and projection methods for their solution.

VIII. Systems of equations. These may be thought of as equations whose kernels are matrix-valued functions. A fundamental problem is the representation of an $n \times n$ matrix-valued function $A(\zeta)$ on the unit circle in the form

$$A(\zeta) = A_-(\zeta)D(\zeta)A_+(\zeta)$$

where $A_-(\zeta)$ and $A_+(\zeta)$ continue to be analytic invertible matrix-valued functions in the regions $|\zeta|>1$ and $|\zeta|<1$ respectively and $D(\zeta)$ is a diagonal matrix each of whose diagonal entries is an integral power of ζ. Such a factorization is shown to be possible in the context of decomposing R-algebras and this is applied to the inversion problem for Toeplitz and Wiener-Hopf operators with matrix-valued symbols. As for the applicability of the natural projection methods, the situation is different than in the scalar case. For example in order that the projection method be applicable to the infinite Toeplitz matrix A with continuous symbol $A(\zeta)$ it is necessary (and sufficient) that both A and its block transpose be invertible. In case $A(\zeta)$ belongs to a decomposing R-algebra this is equivalent to the possibility of both a right factorization of $A(\zeta)$ in the form $A_-(\zeta)A_+(\zeta)$ and a left factorization in which the roles of $A_-(\zeta)$ and $A_+(\zeta)$ are interchanged.

The last chapter is an appendix on the asymptotics of solutions of homogeneous convolution equations such as the original Wiener-Hopf equation

$$\int_0^\infty k(t-s)\varphi(s)\,ds = \varphi(t) \qquad (0 < t < \infty).$$

A final section gives detailed references to the literature on the subject (which was obviously not done in this review).

This outline of the contents of the book presented only the highlights. There is a vast amount of information here, and the book must be considered indispensable to all workers in the field. It is written in the same clear style as the earlier books by Gohberg and M. G. Kreĭn, *Introduction to the theory of linear nonselfadjoint operators* and *Theory and applications of Volterra operators on Hilbert space*, which are volumes 18 and 24 respectively of this series of American Mathematical Society Translations. Very little prior knowledge is required beyond elementary functional analysis and the book can therefore profitably be used by the nonexpert who wants to learn about convolution operators.

HAROLD WIDOM