Values of non-atomic games, by R. J. Aumann and L. S. Shapley, Princeton University Press, Princeton, New Jersey, 1974, xi+333 pp., $14.50

Nonatomic games are games played not by a set of individuals but by a measurable space whose measurable sets are called *coalitions*. They are intended as models for economic problems in large populations. Evidently a case could be made—though neither the book nor this review proposes to—that nonatomic games are more fundamental for economic theory than n-player games. No doubt, actual populations are finite; but that is true also of the atoms in a continuous medium.

This is the first book on nonatomic games, and all of the literature in the area is within the conceptual framework that Aumann and Shapley have established. In particular, it is all on values. As the authors say, "an operator that assigns to each player of a game a number that purports to represent what he would be willing to pay to participate $\cdots$ is called a *value*. Value theory for finite games—i.e., n-player games with n finite—was first studied by Shapley [12], and is by now a well established branch of game theory. It is the purpose of this book to develop a corresponding theory for nonatomic games."

The fact is, thirty years after the first book on n-player game theory [10], that none of its major concepts except that of value seems fit to extend to infinity. The Shapley value is wholly noncontroversial—that is, if we understand it in a suitably narrow sense. Aumann and Shapley do stay within that sense in this book. It is as follows. The process of finding a value for a game is commonly split into two stages: finding a characteristic function v, and passing on to a value φv. When Shapley introduced the operator $v \mapsto \varphi v$ [12], the von Neumann-Morgenstern definition of characteristic function [10] was the only one available. As the literature on values grew, Shapley published a footnote [14] saying that Harsanyi's characteristic function ([3] or [4]) "is to be preferred" in valuation theory. Not all agree [8]. But every two-stage evaluation of this type follows Shapley from v to φv, and the principal one-stage evaluation [11] agrees in its conclusion with Harsanyi and Shapley—it is intended only to deepen the theoretical basis.

In this book, the game is given as a characteristic function, almost always a real-valued function of bounded variation on the Borel sets of an interval (or an isomorphic measurable space I). Evaluating such a function is the same sort of problem as integrating a suitable function, or for a closer parallel, associating a measure to an outer measure. The initial source of ideas is different, and of course one may profitably revisit the source, but the source does not surround the theory

As for the major concepts of game theory untouched in this book, of course lack of unanimity and other difficulties do not forbid generalization. Since one tends to look to game theory for statements about how a game may come out as well as about how much the seats should sell for, it seems worth noticing some difficult features of, say, von Neumann-Morgenstern solution theory. A solution is a set of outcomes; compact, but not otherwise restricted in general [13]. A finite game usually has infinitely many solutions, but sometimes none [9]. It is a familiar complaint of interpreters of game theory experiments that one cannot tell whether the results confirm or disconfirm solution theory. What it means for the players of a game to adopt a solution was explained by von Neumann and Morgenstern [10]; it is rather like adopting a creed, such as single-taxism. Then of course there are the alternative "outcome" theories.

There are four commonly recognized bases, or justifications, for the (Harsanyi-) Shapley value in finite games. (a) It is the only linear operator from games to measures which is invariant under all permutations of players and satisfies two further simple axioms. (b) It is the expected result of totally ordering the players at random and crediting each player with the increment of strength he brings to the coalition consisting of his predecessors. (c) Harsanyi's analysis [3], [4] is not expressed as a construction of v and passage (with Shapley) on to φv, but as a construction of a certain bargaining model which is then treated by Zeuthen-Nash theory. (d) Finally, there is Selten's derivation [11] going from the rules of the game to φv by means of nine axioms.

The first half of this book develops three approaches to evaluating nonatomic games, based on analyses (a) and (b) and on approximation by finite games. (Basing an approach on (c) or (d) would be a far more complex task, and the spirit would have to be more combative since (c) and (d) for finite games are not undisputed.) The results are three value operators defined on certain Banach spaces of games on a standard Borel space; they agree on intersections in at least two of the three cases. All cover the Banach space pNA of set functions spanned by C^1 functions of finite numbers of countably additive, purely nonatomic, totally finite measures. (The C^1 function f must vanish at the origin; then $f(\mu_1, \cdots, \mu_n)$ has finite total variation, which gives the norm used.)

An example of a game in pNA is $\sin \lambda$ where λ is Lebesgue measure (on $[0, 1]$, the usual model of the standard space). The play of this game is a bit curious. If ten small coalitions of measure 0.1 form, each "commands" $\sin 0.1 \approx 0.10$. They cannot all get it, since the total coalition I is worth only $v(I) = \sin 1 \approx 0.84$. But this theory does not concern play. The value of $\sin \lambda$ is $(\sin 1)\lambda$.

Nearly a third of the book concerns a type of Walrasian economy. Very roughly, the central theme of a sizable literature is that several analyses, which in general give different and separately nonunique results, tend to the same single point in suitable large economies. That is the main theme of W. Hildenbrand's new book [5], which does not have Aumann-Shapley values in it. Aumann and Shapley are here concerned with maximal theorems asserting that the core is a single point equal to the (or: to a) value. They give a much sketchier treatment of Walras equilibria. This does give the mathematical reader the connection, and one could pursue the topic in [5].

The axiomatic analysis (a) occupies Chapter I. A reader prepared by a sound introductory real-variable course should have no real difficulty except the one Aumann and Shapley create by omitting a nine-page appendix from the preliminary edition of the book (published as a technical report by both authors' home institutions [2]). They invite the reader to reproduce it, to establish Lemma 8.5.

A *value* is defined as a linear map φ whose domain is a linear subspace of the space of functions of bounded variation, invariant under all automorphisms of the measurable space I, whose codomain is the space of bounded finitely additive measures, and which is invariant, positive on monotone functions, and efficient: $(\varphi v)(I)=v(I)$. There is, of course, no value defined at a game v which is 1 on I (or, on complements of countable sets) and 0 everywhere else. Avoiding that sort of discontinuity, the authors introduce the closed span of the functions of measures $f(\mu)$, μ a nonatomic probability measure and f a b.v. function continuous at 0 and 1 with $f(0)=0$; on this space, there exists a unique value φ. Moreover, all measures φv are countably additive. (And purely nonatomic. The domain of φ is a large space, containing pNA, but in this chapter there are no atoms. Still, a finite game can be lifted up by replacing players by intervals.)

In Chapter II the authors first show the impossibility of carrying over analysis (b) directly by means of a probability measure ω on the space of total orderings of I, even for the best-behaved space pNA. Indeed, with a reformulation putting the invariance on ω instead of on φ, the square of Lebesgue measure cannot be evaluated in this way. Instead (therefore) of averaging evaluations given by all orderings $\mathscr{R}$ of I, they take the limit of evaluations given by orderings $\theta_n\mathscr{R}$, using a mixing sequence θ_n. A full statement would perhaps be excessive here; but the quantifiers in the definition of the space MIX of games v say that for some nonatomic probability measure μ_v, for every measurable order $\mathscr{R}$, for every measure μ with respect to which μ_v is a.c. and μ-mixing sequence $\{\theta_n\}$, the evaluations of v given by $\theta_n\mathscr{R}$ converge to a limit called (again) φv depending only on v. The theorem is that MIX is a closed linear subspace containing pNA and φ is a value on MIX.

Chapter III proceeds straightforwardly with suitably fine finite partitions of I, getting a value on another space of games ASYMP. The authors show that (only) now one has a value for the Middle Eastern game v in which $v([0, \frac{1}{2}])=v([\frac{1}{2}, 1])=\frac{1}{2}$, and v is Lebesgue measure on subsets of those sets, but "left" and "right" players fight when brought together, so that the strength of a mixed coalition is just the measure of the excess of left or right players in it. (Those who can find nothing to do but work.) The value is identically 0, of course.

That example is really a difficult game, not only for its players but for its evaluator. If the political boundary is moved from the midpoint $\frac{1}{2}$, the asymptotic value jumps to become λ on the majority side, $-\lambda$ on the minority side. This is a large change of the game, in the variation norm. Indeed, all three values are operators of norm 1.

Chapter IV introduces fuzzy sets; precisely, the set $\mathscr{I}$ of measurable functions from I to $[0, 1]$, called *ideal sets*. The need for nonreal sets appeared in Chapter II, where the authors pointed out that a "random" coalition T would be independent of any fixed coalition S ($\mu(S\cap T)=\mu(S)\mu(T)$ for any fixed probability measure). Also, the construction of the unique value φ in Chapter I is related to a formula for those v of the form $f(\mu_1, \cdots, \mu_n)$:

$$(\varphi v)(S) = \int_0^1 f_S(t\mu_1(I), \cdots, t\mu_n(I))\, dt,$$

where f_S is the directional derivative of f in the direction $(\mu_1(S), \cdots, \mu_n(S))$. Now the authors produce a distinguished linear operator extending each v in pNA to real-valued v^* defined on ideal sets. They show that $v_S(t)=dv^*(tI+\tau S)/d\tau$ (at $\tau=0$) exists a.e. (overwhelmingly: for almost all t, for all S) and φv is its integral.

Technically, the extension v^* is essential for the generality of the results on cores. In case v is a nonatomic countably additive measure, it is integration, $v^*(f)=\int f\, dv$. The rest is shown to be determined by requiring linearity, multiplicativity, and preserving monotonicity. The essence of the basic core theorem (the *core* of a game v is the set of measures which are $\geqq v$ with equality at I, i.e. which distribute the loot so that every coalition gets as much as it can claim) is in the case of superadditive $f(\mu_1, \cdots, \mu_n)$ with f a C^1 function which is homogeneous of degree 1. For such a game, the core is shown to be the singleton of the value.

The economies considered are more simply described in terms of production than in (more usual) terms of exchange. There are n kinds of raw material and only one kind of finished good. There is a fixed probability measure μ on I, and a vector-valued raw material density $\mathbf{a}$; $[s, s+ds]$ begins with $\mathbf{a}(s)\mu(ds)$. There is a technology $u=u(x, s)$. Coalition S could blindly set to and produce $\int_S u(\mathbf{a}(s), s)\, d\mu(s)$, but if S acts cooperatively it will first redistribute its supply $\mathbf{a}$ to some other $\boldsymbol{x}$—the same amount of material, $\int_S x\, d\mu = \int_S \mathbf{a}\, d\mu$—to get maximum production $v(S)$. The broadest theorem is that if $u(x, s) = o(\|x\|)$ as $\|x\| \to \infty$ (diminishing returns), μ-integrably in s, and smoothness prevails ($\mathbf{a}$ integrable, $u(\ , s)$ C^1 interior to the positive orthant and continuous at the boundary, u Borel-measurable), then $v \in \mathrm{MIX} \cap \mathrm{ASYMP}$ and the core is the singleton of the (mixing=asymptotic) value. If $\mathbf{a}$ is strictly positive or there are only finitely many types of technology $u(\ , s)$, then $v \in pNA$.

To follow the treatment of the economic models, one must add Aumann and Perles' paper [1] to the book; it is cited for some basic things such as the existence of v.

The treatment of cores and economies occupies Chapters V, VI, and the latter part of VII. The one-and-a-half chapters not yet described are quite good—had the book ended here and the rest appeared as two papers, they would be two big steps forward—but of much narrower interest. Let us turn to other aspects for a bit.

The only alternative to the Harsanyi-Shapley evaluation for finite games of this type (i.e., with unrestricted side payments) in the literature is the reviewer's [**8**]. It is true, but somewhat misleading, to say that it differs only in constructing a different v and then using the same operator $v \mapsto \varphi v$. That is conceptually wrong since the justification is a bargaining model patterned after Harsanyi's. And technically, the characteristic function v obtained is not only different for particular games but in a different space, for v in [**8**] (v-; a construct of Harsanyi's, used by him for something else) is always strong. The axiomatic justification (a) for Shapley value $v \mapsto \varphi v$ relativizes to strong v [**6**], and of course (b) or any other constructive procedure relativizes to any subclass. Nothing is known about nonatomic extensions of the value of [**8**]. (Of course, it is as hard a problem as the extension of Harsanyi's procedure.)

Aumann and Shapley go far to avoid controversy, and let it be avoided (here), but I think they go so far as to mislead the reader. They describe their subject as games with unrestricted side payments and fixed threats. This is the customary expansive way of saying "games defined by v", but it ought not to be. A probability mixture of games with actually fixed threats does not have fixed threats (this matters for $0.8\Gamma_{45}+0.2\Gamma_{90}$ in [**8**], contrasted with Γ_{54} which really has fixed threats). So anyone treating a whole Banach space of games is not really discussing fixed-threat games.

Aumann and Shapley do not cite the reviewer or Selten, but they cite Harsanyi, unfortunately only for the 1959 version of his model [**3**]. That was superseded by the later version [**4**]. We are assured that the revision was not prompted by the reviewer's criticism. Still, if Aumann and Shapley really mean to resurrect a theory of seat prices which sells for 9 units the right to play in a game in which one can certainly gain 10 units, no matter what the other players do (not a game of the type treated in this book, but nothing in the game involves subtraction [7]), they ought to offer some statement about it.

Now if we consider the whole 358-page book consisting of the book under review, Appendix A of [2], and [1], it is superb exposition. Definitions and results are clearly labeled. Definitions can be found, via the 11-page general index and the page of names (mostly of subspaces of BV). Precise and imprecise discussions are clearly separated, and both are given when needed. Many notes describe related results, and some instructive ideas for simplification or extension that do not work. In four of the chapters, the main results are described in a section after the first, and for good reasons. I suspect that the six ways suggested to read only selected parts of the book will not work very well, but they could work in principle. Reading the whole book, one has practically all the nonatomic game theory that now exists. One may hope that in a few years there will be other books giving much more material. Even then, beginners will surely make good use of Aumann-Shapley, since there is no reasonable prospect of its successors' being so well written.

Physically, the book sustains the high standards of the Princeton Press.

The value theory, we have seen, is uniquely determined by the axioms on a space larger than pNA; on pNA, it is given also by mixing transformations (or sequences), by finite approximations, or by a formula $\int v_S(t)\,dt$. Two of the constructive approaches carry further, over MIX, ASYMP respectively. It is not known if they are consistent on MIX $\cap$ ASYMP.

The further results are less simply stated. The most convincing showing that they extend a single theory (and are not just artificial extreme extensions of techniques) is the application already stated, the expected result for Walrasian economies with artificial assumptions (strict positiveness, finite type) removed. More broadly, these results turn on the same ideas taken in weaker senses. The possibilities seem by no means exhausted. For a frivolous illustration, one might try generalizing greatly the C^1 functions at the heart of pNA and using derivatives in the sense of distribution theory.

The third construction, the integral formula, also extends further. For this to be possible (using Newton-Leibniz derivatives), one must first have an operator $v \mapsto v^*$ beyond pNA. That is gotten by observing that $v \mapsto v^*$ is continuous even in the sup norm. As it is linear, it therefore extends over the sup-norm closure of pNA; and the new operator is shown to retain the basic properties.

The next idea (the keynote of Chapter VII) is motivated most simply by inspecting the integral formula. It determines (on pNA) φv from derivatives of v^* at ideal sets tI only, i.e. along the diagonal. Accordingly φv is determined by the values of v^* on any neighborhood of the diagonal. The authors abstract this "diagonal property" in terms of v (not v^*) as follows: $v \in \mathrm{DIAG}$ if there is a k-tuple ζ of nonatomic probability measures such that v vanishes at every coalition taken by ζ into some (fixed) neighborhood of the diagonal in k-space. All values yet constructed vanish wherever defined in DIAG. Adding this to the definition of value—for the definition of *diagonal value*—one gets an obvious extension of the uniqueness theorem. Further, $pNA+\mathrm{DIAG}\subset\mathrm{ASYMP}$. On the intersection of the variation-norm closure of $pNA+\mathrm{DIAG}$ and the sup-norm closure of pNA, the integral formula gives the unique (and asymptotic) diagonal value. In the same space, the core theorem holds under appropriate superadditivity and homogeneity conditions. Only a somewhat smaller space is shown to be contained in MIX, but enough for the Walrasian economies.

As suggested earlier, Aumann and Shapley see these operators as worth having apart from any connection with games. In this spirit, they conclude with a short chapter on "games" on a nonstandard measurable space. All three initial approaches fail here for lack of measurable automorphisms, mixing sequences and sequences of finite partitions. Yet pNA and $v\mapsto v^*$ stand like a stone wall. Everything not involving MIX or ASYMP is restored if the invariance axiom is replaced by a normalization condition: under suitable restrictions, $\varphi(f(\mu))=\mu$. That is substantially all, though the asymptotic theory holds over subspaces of a standard space.

J. R. ISBELL

REFERENCES

1. R. J. Aumann and M. Perles, *A variational problem arising in economics*, J. Math. Anal. Appl. **11** (1965), 488–503. MR **31** #6670.

2. R. J. Aumann and L. S. Shapley, *Values of non-atomic games*. Part I: *The axiomatic approach*, RM-5468-PR, The Rand Corporation, Santa Monica; also RM-42, The Hebrew University of Jerusalem, 1968.

3. J. C. Harsanyi, *A bargaining model for the cooperative n-person game*, Contributions to the Theory of Games, vol. 4, Princeton Univ. Press, Princeton, N.J., 1959, pp. 325–355. MR **21** #4062.

4. ———, *A simplified bargaining model for the n-person cooperative game*, Internat. Economic Review **4** (1963), 194–220.

5. W. Hildenbrand, *Core and equilibria of a large economy*, Princeton Univ. Press, Princeton, N.J., 1974.

6. J. R. Isbell, *Absolute games*, Contributions to the Theory of Games, vol. 4, Princeton Univ. Press, Princeton, N.J., 1959, pp. 357–396. MR **24** #B412.

7. ———, *A modification of Harsanyi's bargaining model*, Bull. Amer. Math. Soc. **66** (1960), 70–73. MR **22** #13315.

8. ———, *Contract law and the value of a game*, Israel J. Math. **5** (1967), 135–144. MR **39** #7947.

9. W. F. Lucas, *The proof that a game may not have a solution*, Trans. Amer. Math Soc. **137** (1969), 219–229. MR **38** #5474.

10. J. von Neumann and O. Morgenstern, *Theory of games and economic behavior*, Princeton Univ. Press, Princeton, N.J., 1944. MR **6**, 235.

11. R. Selten, *Valuation of n-person games*, Advances in Game Theory, Princeton, N.J., 1964, pp. 577–626. MR **29** #1081.

12. L. S. Shapley, *A value for n-person games*, Contributions to the Theory of Games, vol. 2, Princeton Univ. Press, Princeton, N.J., 1953, pp. 307–317. MR **14**, 779.

13. ———, *A solution containing an arbitrary closed component*, Contributions to the Theory of Games, vol. 4, Princeton Univ. Press, Princeton, N.J., 1959, pp. 87–94. MR **22** #635.

14. ———, *Values of large market games: Status of the problem*, RM-3957-PR, The Rand Corporation, Santa Monica, 1964.

Convolution equations and projection methods for their solution, by I. C. Gohberg and I. A. Fel'dman, American Mathematical Society Translations, vol. 41, 1974, ix+261 pp.

Suppose $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are Banach spaces, $\{P_\tau\}$ and $\{Q_\tau\}$ are families of projection operators on $\mathfrak{B}_1$ and $\mathfrak{B}_2$ respectively which converge strongly as $\tau\to\infty$ to the respective identity operators, and A is a bounded linear transformation from $\mathfrak{B}_1$ to $\mathfrak{B}_2$. One says that the projection method (P_τ, Q_τ) is applicable to A if, roughly speaking, $(Q_\tau AP_\tau)^{-1}$ converges strongly to A^{-1} as $\tau\to\infty$. More precisely what is required is that $Q_\tau AP_\tau$, as an operator from $P_\tau\mathfrak{B}_1$ to $Q_\tau\mathfrak{B}_2$, be invertible for sufficiently large τ and that $(Q_\tau AP_\tau)^{-1}Q_\tau$ converge strongly as $\tau\to\infty$. (Then A is necessarily invertible and the strong limit is A^{-1}.)

To give an example, the prototype of those considered in this book, let a be a bounded function defined on the unit circle having Fourier coefficients a_k ($k=0, \pm 1, \cdots$), and consider the operator A on l_2 of the positive integers defined by

$$A\{\xi_j\} = \left\{\sum_{k=1}^{\infty} a_{j-k}\xi_k\right\}_{j=1}^{\infty}.$$

This is the (semi-infinite) Toeplitz operator associated with a. The projections are the simplest ones: $P_n=Q_n=$projection on the subspace of sequences $\{\xi_j\}$ satisfying $\xi_j=0$ for $j>n$. The operator P_nAP_n may then be represented by the finite Toeplitz matrix

$$A_n = (a_{j-k})_{j,k=1}^{n}$$

and the question is whether these matrices are invertible from some n onward and, if so, whether the inverses of these matrices converge strongly