BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 2, March 1975

MEASURES AS CONVOLUTION OPERATORS ON HARDY AND LIPSCHITZ SPACES

BY MISHA ZAFRAN[1]

Communicated by Alberto Calderón, November 11, 1974

In this note we announce some new results concerning the spectral theory of measures as convolution operators. To state our principal theorem, we introduce the following notation. If X is a Banach space and T is a bounded linear operator on X, we write $\mathrm{sp}(T, X)$ to denote the spectrum of T on X. Let G be an LCA group with dual group Γ. $M(G)$ will denote the class of finite regular Borel measures on G, and $M_0(G) = \{\mu \in M(G) \mid \hat{\mu}$ vanishes at infinity on $\Gamma\}$. For $\mu \in M(G)$, let T_μ denote the operator defined by $T_\mu(f) = \mu * f$, that is, convolutions with μ. Finally, let H^1 be the natural domain of the Hilbert transform on $L_1(\mathbf{R})$, and let Lip α denote the usual class of bounded functions on $\mathbf{R}$ satisfying a Lipschitz condition of order α, $0 < \alpha < 1$. We can now state our main result.

THEOREM 1. *There exists a measure* $\mu \in M_0(\mathbf{R})$ *such that*
(a) $\mathrm{sp}(T_\mu, H^1) \neq \hat{\mu}(\mathbf{R}) \cup \{0\}$, *and*
(b) $\mathrm{sp}(T_\mu, \mathrm{Lip}\,\alpha) \neq \hat{\mu}(\mathbf{R}) \cup \{0\}$, $0 < \alpha < 1$.

This may be viewed as an analogue of the now classical Wiener-Pitt theorem concerning the invertibility of Fourier-Stieltjes transforms [4, Theorem 5.3.4]. Moreover, an elementary interpolation argument shows that if $1 < p < \infty$,

$$\mathrm{sp}(T_\nu, L_p) = \hat{\nu}(\mathbf{R}) \cup \{0\},$$

for all $\nu \in M_0(\mathbf{R})$ (see [1, §1.4]). Thus, in a sense, our theorem is intermediate between the L_1 and L_p $(1 < p < \infty)$ cases.

The proof of Theorem 1 is based on the following result.

THEOREM 2. *Let*

AMS (MOS) subject classifications (1970). Primary 43A32, 47B99; Secondary 42A18.

[1]This work was supported in part by NSF grant GP 28112A.

$$\psi_\delta(x) = \begin{cases} 1/\delta & \text{if } 0 \leq x < \delta/2, \\ -1/\delta & \text{if } -\delta/2 < x < 0. \end{cases}$$

Then there exists a measure $\nu \in M_0(\mathbf{R})$, *of total variation norm* 1, *which satisfies*

$$(*) \qquad \limsup_{\delta \to 0} \| \nu^j * \psi_\delta \|_1 = 1,$$

for all $j = 1, 2, \cdots$.

The expression (*) arises since, as is readily verified,

$$\limsup_{\delta \to 0} \| T(\psi_\delta) \|_1 \leq C \| T \|_{O(H^1)}$$

for every bounded linear operator T on H^1. Here C is an absolute constant and $\|T\|_{O(H^1)}$ denotes the operator norm of T on H^1. Moreover, if $g \in L_1(\mathbf{R})$, $\lim_{\delta \to 0} \|g * \psi_\delta\|_1 = 0$. Therefore, the expression (*) provides us with a lower bound for the norms $\|(T_\nu - T_g)^j\|_{O(H^1)}$, $j = 1, 2, \ldots$, for every $g \in L_1(\mathbf{R})$. Consequently, since $\hat{\nu}$ vanishes at infinity, we have for appropriate $f \in L_1(\mathbf{R})$ that $\|\widehat{\nu - f}\|_{L_\infty(\mathbf{R})} < 1$, whereas the spectral radius of the operator $T_{\nu - f}$ on H^1 is at least 1.

A similar argument also applies to the space Lip α and certain of its variants, specifically, certain of the Taibleson spaces (see [5]). Thus Theorem 1 follows from Theorem 2 (with $\mu = \nu - f$).

The objects of study on Theorem 2 are measures of Cantor-Lebesgue type, which are subject to certain arithmetic constraints. Specifically, we examine infinite Bernoulli convolutions of the form

$$\nu = \mathop{*}_{k=1}^{\infty} (\tfrac{1}{2}\delta_0 + \tfrac{1}{2}\delta_{t_k}),$$

where the positive sequence $\{t_k\} \in l_1$ is chosen so that

(1) $t_{k+1}/t_k \longrightarrow 0$ as $k \longrightarrow \infty$, and $t_n > \Sigma_{k=n+1}^{\infty} t_k$, $n = 1, 2, \ldots$,

(2) $\hat{\nu}$ vanishes at infinity on $\mathbf{R}$, and

(3) $\{t_k\}$ is fully independent, that is, if $\{n_k\}$ is any bounded sequence of integers, and if $\Sigma_{k=1}^{\infty} n_k t_k = 0$, then $n_k = 0$, $k = 1, 2, \cdots$.

The existence of such sequences is guaranteed by probabilistic considerations (see [3, pp. 256–258]).

The proof of Theorem 2 then consists largely of a careful study of the j-fold sum of the Cantor set $\{\Sigma_{k=1}^{\infty} \epsilon_k t_k \mid \epsilon_k = 0 \text{ or } 1\}$ generated by sequences

$\{t_k\}$ of the above form. In particular, we show that the j-fold sum itself "looks like" a Cantor-type set which has been constructed in a "regular" way. We then integrate along the gaps arising at the various stages of the construction, to obtain estimate (*) in Theorem 2.

Finally, we remark that the techniques used here also yield the analogue of Theorem 1 for the circle group. Further results, detailed proofs, and some applications of this theory will appear in [6].

REFERENCES

1. L. Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta Math. 104 (1960), 93–140. MR 22 #12389.

2. J.-P. Kahane and R. Salem, *Ensembles parfaits et séries trigonometriques*, Actualités Sci. Indust., no. 1301, Hermann, Paris, 1963. MR 28 #3279.

3. Y. Meyer, *Algebraic numbers and harmonic analysis*, North-Holland, Amsterdam, 1972.

4. W. Rudin, *Fourier analysis on groups*, Interscience Tracts in Pure and Appl. Math., 12, Interscience, New York, 1962. MR 27 #2808.

5. M. H. Taibleson, *On the theory of Lipschitz spaces of distributions on Euclidean n-space. I. Principal properties*; II, *Translation invariant operators, duality, and interpolation*, J. Math. Mech. 13 (1964), 407–479; ibid. 14 (1965), 821–839. MR 29 #462; 31 #5087.

6. M. Zafran, *Measures as convolution operators on H^1 and* Lip α (submitted).

DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY, STANFORD, CALIFORNIA 94305