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REAL FORMS OF HERMITIAN SYMMETRIC SPACES[1]

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Introduction. The results we give here only begin to answer the following general problems: Let X be a hermitian symmetric domain, Γ a group acting holomorphically and discontinuously, and $U = \Gamma\backslash X$ the quotient. Then, by Kodaira if U is compact and smooth, or, by Baily-Borel if just U has finite volume (and Γ arithmetic), U is algebraic. One can ask for all ways of algebracizing U over $\mathbf{R}$, for each of the number of connected components of $U(\mathbf{R})$, and the type of each component as, say, a real analytic space. For the smooth $\mathbf{R}$-algebraic varieties $U = \Gamma\backslash X$, each component of $U(\mathbf{R})$ is a quotient of the form $U' = \Gamma'\backslash X'$ of a globally symmetric space $X' \subset X$ by a subgroup $\Gamma' \subset \Gamma$. To determine which X' and Γ'-actions occur is our goal.

Generalities. Let X be as above, $\sigma\colon X \to X$ an antiholomorphic involution, and X' the set of fixed points of σ. (We consider Γ only virtually now.)

PROPOSITION. (a) *σ is an isometry of the Bergmann metric.*

(b) *X' is a nonempty connected totally geodesic subsymmetric space of X; $\dim_{\mathbf{R}} X' = \dim_{\mathbf{C}} X$.*

(c) *X' is holomorphically dense: a holomorphic or antiholomorphic automorphism of X is determined by its restriction to X'.*

One can construct, at least for X without "exceptional" factors, (for example via the Lie algebra of the isometry group) involutions as above. Choose σ_0 as "standard" and x_0 a fixed point of σ_0. Let G^h be the group of holomorphic automorphisms of X, K^h the isotropy group at x_0. Then Gal $= \{1, \sigma_0\}$ acts by conjugation on G^h and K^h. Moreover if C denotes the set of all antiholomorphic involutions of X, and C_0 the subset fixing x_0, then G^h acts by conjugation on C and K^h preserves C_0. The quotients C/G^h and C_0/K^h are the G^h- and K^h-conjugacy classes of C and C_0. These can be

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identified with the cohomology sets $H^1(\mathrm{Gal}, G^h)$ and $H^1(\mathrm{Gal}, K^h)$.

PROPOSITION. *The canonical map* $H^1(\mathrm{Gal}, K^h) \to H^1(\mathrm{Gal}, G^h)$ *induced by the inclusion of* K^h *in* G^h *is a bijection.*

One wishes to compute $H^1(\mathrm{Gal}, G^h)$, and from it the isometry types of real forms $X' \subset X$ of antiholomorphic involutions. The standard maps σ_0 can be chosen so that the crucial fact becomes:

LEMMA. $H^1(\mathrm{Gal}, U(n)/\{\pm 1\})$ *with action = complex conjugation is trivial for n odd, and for n even has two elements represented by the identity matrix and* $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Results. Since all of our X and X' are globally symmetric of purely noncompact type, we can (if we are interested only in isometry types) refer to them by the Lie algebra of the isometry group. Denote by $c(X)$, $\mathrm{card}(H^1(\mathrm{Gal}, G^h))$, and call it the "number of complex conjugations of X". The results below show that different conjugations have real forms with distinct isometry types.

X		$c(X)$	X'
$\mathfrak{sp}(n, \mathbf{R})$ –	n odd	1	$\mathbf{R} \times \mathfrak{sl}(n, \mathbf{R})$
	n even	2	$\mathbf{R} \times \mathfrak{sl}(n, \mathbf{R})$, $\mathfrak{sp}(n/2, \mathbf{C})$
$\mathfrak{so}^*(2n)$ –	n odd	1	$\mathfrak{so}(n, \mathbf{C})$
	n even	2	$\mathfrak{so}(n, \mathbf{C})$, $\mathbf{R} \times \mathfrak{su}^*(n)$
$\mathfrak{so}(p, 2)$ –	p odd	$(p + 1)/2$	$\mathfrak{so}(k, 1) \times \mathfrak{so}(p - k, 1)$, $0 \leqslant k < p/2$
	p even > 2	$(p/2) + 1$	$\mathfrak{so}(k, 1) \times \mathfrak{so}(p - k, 1)$, $0 \leqslant k \leqslant p/2$
$\mathfrak{su}(p, q)$ –	$p \neq q$ or $p = q = 1$		
	not both even	1	$\mathfrak{so}(p, q)$
	both even	2	$\mathfrak{so}(p, q)$, $\mathfrak{sp}(p/2, q/2)$
	$p = q > 1$		
	odd	2	$\mathfrak{so}(p, p)$, $\mathbf{R} \times \mathfrak{sl}(p, \mathbf{C})$
	even	3	$\mathfrak{so}(p, p)$, $\mathbf{R} \times \mathfrak{sl}(p, \mathbf{C})$, $\mathfrak{sp}(p/2, p/2)$

If X is a product of irreducibles (all nonexceptional), we can describe the conjugations of X in terms of the conjugations of the factors.

THEOREM. (a) $c(X)$ *is finite.*

(b) *If* $X = \Pi_i X_i^{n(i)}$, *where the* X_i *are nonisomorphic irreducibles, then* $c(X) = \Pi_i c(X_i)^{n(i)}$.

(c) *If X is irreducible,* $c(X^n) = \Sigma_{l \equiv n\ (2); 0 \leqslant l \leqslant n}\, c(X)^{l/}$, *where the exponent denotes "symmetric" lth power of the cardinality* $c(X)$.

REMARKS. To illustrate (c) in the above theorem, we give a "typical" conjugation of X^5 if $c(X) = 2$.

$$(x, y, z, u, v) \mapsto (a(y), a^{-1}(x), \sigma_0(z), \sigma_0(u), \sigma_1(v))$$

where a denotes any antiholomorphic automorphism, and the ordering σ_0, σ_0, σ_1 is irrelevant. The corresponding X' is $X \times X_0' \times X_0' \times X_1'$. Part (c) of the Theorem and the above table show that there are symmetric spaces which do not occur as X', for example $\mathfrak{sl}(n, \mathbf{R})$ and $\mathfrak{sl}(n, \mathbf{C})$. Finally, almost all of the conjugations in the table have been interpreted either geometrically on X, or as elements of classical matrix groups acting on X.

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