BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 2, March 1975

THE STABILITY PROBLEM IN SHAPE AND A WHITEHEAD THEOREM IN PRO-HOMOTOPY

BY DAVID A. EDWARDS AND ROSS GEOGHEGAN[1]

Communicated by P. T. Church, November 27, 1974

1. **Shape.** We give a solution to the following

PROBLEM. *Give necessary and sufficient conditions for a compactum Z to have the shape of* (A) *a complex or* (B) *a finite complex.*

Problem B makes sense in Borsuk's shape theory for compacta [2] but in order to give meaning to Problem A, we must extend Borsuk's theory to include noncompact complexes. A particularly simple treatment is in [7]. Alternatively one can replace "complex" by "ANR" in Problem A, and use Fox's extension to metric spaces [9].

It is desirable that the conditions in Problems A and B be intrinsic. The following partial solution to Problem B is in [10]: *a finite-dimensional* 1-*UV compactum has the shape of a finite complex if and only if its Čech cohomology with integer coefficients is finitely generated.* But without the hypothesis 1-*UV*, the condition offered in [10] is not an intrinsic one.

Now for our solution. First some notation. If (Z, z) is a pointed connected compact subset of a euclidean space E, let $\{(X_\alpha, z)\}$ be the inverse system of all connected open neighborhoods of Z in E, pointed by z and bonded by inclusion. Regarding $\{(X_\alpha, z)\}$ as an object of pro-H_0 [1] let pro-$\pi_k(Z, z)$ be the pro-group $\{\pi_k(X_\alpha, z)\}$; let $\check{\pi}_k(Z, z)$ be its inverse limit (the kth shape group of (Z, z)). Let $\widetilde{K}^0(G)$ denote the reduced projective class group of the group G (see p. 64 of [12]).

THEOREM 1 [8]. *Let (Z, z) be as above. The following are equivalent:* (i) *pro-$\pi_k(Z, z)$ is isomorphic to $\check{\pi}_k(Z, z)$ in pro-groups for each $k \geqslant 1$;* (ii) *(Z, z) has the pointed shape of a pointed complex of dimension* $\max\{3, \dim Z\}$; (iii) *(Z, z) is dominated in pointed shape by a pointed finite complex*; (iv) *(Z, z) is movable and the natural topology on $\check{\pi}_k(Z, z)$ is discrete for each $k \geqslant 1$;* (v) *(Z, z) is a pointed FANR. Furthermore, Z has the shape*

AMS (MOS) subject classifications (1970). Primary 55D99.
[1]Supported in part by NSF Grant PO38761.

of a finite complex if and only if an intrinsically defined "Wall obstruction" $w(Z, z) \in \widetilde{K}^0(\check{\pi}_1(Z, z))$ vanishes. All possible Wall obstructions occur among two-dimensional compacta.

Movable is defined in [4]; (iv) is explained in [6]; FANR is defined in [3]. Note that if $\check{\pi}_1(Z, z)$ is free or free abelian, $w(Z, z) = 0$.

2. **Pro-homotopy**. Shape is the "inverse limit" of pro-homotopy and the above results are proved by means of new theorems in pro-homotopy: among them a Whitehead theorem (Theorem 2) and a stability theorem (Theorem 3).

If C is category, let pro-C be the category whose objects are inverse systems in C indexed by directed sets, and whose morphisms are as described in the Appendix to [1]. An object of pro-C indexed by the natural numbers is a *tower* in C. Let CW_0 be the category of pointed connected (CW) complexes and pointed maps; let H_0 be the corresponding homotopy category. We suppress base points. If $X = \{X_\alpha\}$ is in pro-CW_0 or pro-H_0, CW-dim $X = \sup\{\dim X_\alpha\}$; h-dim $X = \inf\{CW\text{-dim } Y | Y$ is isomorphic to X in pro-$H_0\}$. X is *compact* if each X_α is a finite complex. $\pi_k(X)$ is the pro-group $\{\pi_k(X_\alpha)\}$; $\check{\pi}_k(X)$ is its inverse limit group. A *weak equivalence* is a morphism inducing isomorphisms on π_k for all $k \geqslant 1$.

Theorem 2 is an extension of results in [11].

THEOREM 2 [8]. *Let $g: X \longrightarrow Y$ be a morphism of pro-CW_0 and let $n = \max\{1 + CW\text{-dim } X, CW\text{-dim } Y\} < \infty$. Suppose $g_\#: \pi_k(X) \longrightarrow \pi_k(Y)$ is an isomorphism for $k \leqslant n$ and has a right inverse for $k = n + 1$. Then g induces an isomorphism of pro-H_0. If X and Y are towers, g need only be a morphism of pro-H_0.*

Theorem 3 uses Theorem 2 together with [12].

THEOREM 3 [8]. *Let X be a tower in H_0.* (i) *There exist a pointed complex Q and a weak equivalence $q: Q \longrightarrow X$ in pro-CW_0 if and only if $\pi_k(X)$ is isomorphic in pro-groups to $\check{\pi}_k(X)$ for all $k \geqslant 1$. In case* (i) *holds we have*: (ii) *Q can have dimension* $\max\{3, h\text{-dim } X\}$ *and if h-dim $X = 1$, Q can be a bouquet of circles*; (iii) *if CW-dim $X < \infty$, q induces an isomorphism in pro-H_0*; (iv) *if CW-dim $X < \infty$ and X is compact, Q is domained in H_0 by a finite complex, and X is isomorphic to a (pointed) finite complex P if and only if an intrinsically defined "Wall obstruction" $w(X) \in \widetilde{K}^0(\check{\pi}_1(X))$*

vanishes: *if* $w(X) = 0$, $\dim P = \dim Q$; (v) *all possible Wall obstructions occur among towers of CW*-dim 2.

REFERENCES

1. M. Artin and B. Mazur, *Etale homotopy,* Lecture Notes in Math., no. 100, Springer-Verlag, Berlin and New York, 1969. MR **39** #6883.

2. K. Borsuk, *Concerning homotopy properties of compacta,* Fund. Math. **62** (1968), 223–254. MR **37** #4811.

3. ———, *Fundamental retracts and extensions of fundamental sequences,* Fund. Math. **64** (1969), 55–85; errata, ibid.**64** (1969), 375. MR **39** #4841.

4. ———, *Some remarks concerning the shape of pointed compacta,* Fund. Math. **67** (1970), 221–240. MR **42** #1076.

5. A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations,* Lecture Notes in Math., vol. 304, Springer-Verlag, Berlin and New York, 1972.

6. D. A. Edwards and R. Geoghegan, *Compacta weak shape equivalent to* ANR's, Fund. Math. (to appear).

7. ———, *Shapes of complexes, ends of manifolds, homotopy limits and the Wall obstruction* (to appear).

8. ———, *The stability problem in shape and a Whitehead theorem in pro-homotopy,* Trans. Amer. Math. Soc. (to appear).

9. R. H. Fox, *On shape,* Fund. Math. **74** (1972), no. 1, 47–71. MR **45** #5973.

10. R. Geoghegan and R. C. Lacher, *Compacta with the shape of finite complexes,* Fund. Math. (to appear).

11. S. Mardešić, *On the Whitehead theorem in shape theory,* Fund. Math. (to appear).

12. C. T. C. Wall, *Finiteness conditions for CW-complexes,* Ann. of Math. (2) **81** (1965), 55–69. MR **30** #1515.

DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK AT BINGHAMTON, BINGHAMTON, NEW YORK 13901