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ON THE REPRESENTATION AND APPROXIMATION OF A CLASS OF OPERATOR-VALUED ANALYTIC FUNCTIONS

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1. **Main results.** Let $\mathcal{H}$ be a separable Hilbert space, and a, b, real and finite. Suppose that $T(z)$ is, for each $z \notin [a, b]$, a bounded operator-valued analytic function which satisfies the condition

(R) *For each* $\phi \in \mathcal{H}$ *and* $z \notin [a, b]$, $(T(z)\phi, \phi)$ *maps the upper half-plane into itself and the lower half-plane into itself.*

We then call $T(z)$ an R-operator, and the class of all R-operators with the cut $[a, b]$ will be called $\mathcal{W}[a, b]$. The class $\mathcal{W}[a, b]$ generalizes to operators of the class $R[a, b]$ of R-functions, those analytic functions on the cut complex plane which preserve upper and lower half-planes. For any $f(z) \in R[a, b]$, there is a nonnegative number α, a real number β, and a positive measure μ with support on $[a, b]$, such that

$$f(z) = \alpha z + \beta + \int_a^b \frac{d\mu(t)}{t - z} \tag{1.1}$$

holds (cf. Stone [5, p. 573]). This representation links Stieltjes transforms (the integral given above) to a mapping condition. The intent of this note is to state a generalization of this result to R-operators. Results in this direction have been obtained by J. S. Mac Nerney [3] in connection with the Hermitian moment problem and corresponding continued fractions.

THEOREM 1. *If* $T(z) \in \mathcal{W}[a, b]$, *then there is a bounded nonnegative operator* A, *a selfadjoint operator* B *and a strongly countably additive positive operator-valued measure* μ *such that*

$$T(z) = Az + B + \int_a^b \frac{d\mu(t)}{t - z}.$$

If, in addition, the $T(z)$ *form a compact family of operators, then* $\mu(t)$ *is uniformly countably additive.*

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With this construction of the measure $\mu(t) = \mu([0, t])$, which can be constructed to be left continuous, and assuming $A = B = 0$, we obtain several corollaries:

(1) *$T(z)$ is called meromorphic if for each $\phi \in H$, $(T(z)\phi, \phi)$ is a meromorphic function. $T(z)$ is a meromorphic R-operator if and only if the measure μ is purely atomic.*

(2) *If $b < \infty$ and $\mu(b) \leqslant I$, the identity operator, there exists a Hilbert space $H_1 \supseteq H$ and a selfadjoint operator A on H_1, such that on H, $T(z) = P(A - zI)^{-1} + f(z)(I - \mu(b))$, where $f(z)$ is an R-function and P is the orthogonal projection from H_1 onto H.*

Corollary (2) illustrates the close relation of R-operators to resolvents. Also we wish to point out the similarity of our representation theorem to the result in Fillmore [2] that proves the analogous Herglotz representation. To approach the problem from the Naĭmark extension theory, as would be required in the above situation, one must first have the operator-valued measure constructed in Theorem 1.

Perhaps more important than the representation theorem above is the fact that R-operators can be approximated by rational functions which converge uniformly on compact sets off the cut $[a, b]$. The rational approximants to which we refer are the operator-valued analogues of Padé approximants (cf. [6, J. Zinn-Justin]). It is of course no mystery why Padé approximants are used; in the case of Stieltjes series these approximants converge uniformly on compact sets off the cut $[a, b]$. If the R-operator $T(z)$ has the formal power series $\Sigma_{k=0}^{\infty} T_k z^k$, if $P_N(z)$ and $Q_N(z)$ are polynomials of degree N, and if

$$P_N(z)Q_N(z)^{-1} - T(z) = D_{2N+1}z^{2N+1} + \cdots,$$

we say that $P_N Q_N^{-1}$ is the (right) $[N/N]$ Padé approximant to $T(z)$. [6, J. Zinn-Justin] shows that the right, left and mixed Padé approximants are equal if all exist. Precisely then, we state

THEOREM 2. *If $T(z) \in W[a, b]$, then for each N there is an $[N/N]$ Padé approximant of mixed type, $T_N(z)$, which converges uniformly in the norm topology to $T(z)$ on compact subsets of C bounded away from the cut $[a, b]$.*

We remark that the derivation of the Padé approximants comes from a variant of Nuttall's compact formula (cf. [1, Baker]).

2. **Applications.** The above results can be applied to both represent and approximate the "reaction matrix" originally studied by E. P. Wigner. For a modern account of this theory see [4].

As a second application, consider the Schrödinger time-evolution operator $U(t, z)$ which satisfies

$$i(d/dt)U(t, z) = (H_0 - zV(t))U(t, z), \qquad U(0, z) = I, \tag{2.1}$$

where H_0 is an unperturbed Hamiltonian and $V(t)$ is a bounded uniformly positive definite potential. Although $U(t, z)$ is not an R-operator, the operator $T(t, z) = i(I - U)(I + U)^{-1}$ is. If we let T_N be the $[N/N]$ Padé approximant of T and U_N that of U, then $U_N(t, z) = -(T_N - i)(T_N + i)^{-1}$ (cf. [6, J. Zinn-Justin]). Thus, by Theorem 2, U_N and T_N have roughly the same convergence properties. An essential property of $U(t, z)$ is that it is unitary on the real axis. The Padé approximants preserve this property, whereas the partial sums do not.

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