BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 2, March 1975

BOOK REVIEW

Conjugacy classes in algebraic groups, by Robert Steinberg, Springer-Verlag, New York, 1974, vi+159 pp., $7.00

Suppose A and B are invertible $n\times n$ complex matrices. How can one tell if A and B are conjugate—that is, if there is an invertible $n\times n$ complex matrix C such that $B=CAC^{-1}$? The solution, of course, is to examine the Jordan canonical form of the two matrices. But now suppose A and B lie in some algebraic subgroup of the group of all invertible $n\times n$ complex matrices—that is, in a subgroup determined by some polynomial conditions on the matrix entries, for example the matrices of determinant one. Now how can one tell if A and B are conjugate in that subgroup?

In more formal language, suppose G is a linear algebraic group over an algebraically closed field k, and let G act on itself by conjugation. The quotient set V of G under this action is the set of conjugacy classes of G. What one really wants to know is the structure of V, or of some subsets of it determined by looking at conjugacy classes of elements of special types. It is to the analysis of this structure that Robert Steinberg's lecture notes *Conjugacy classes in algebraic groups* are directed.

To begin, it is possible to carry out some of the Jordan decomposition in G. Every element x of G, can be written as a product x_sx_u, where x_s acts semisimply on any finite dimensional vector space V on which G acts linearly and algebraically; that is, V has a basis of eigenvalues of x_s, while x_u acts unipotently on V; that is all its eigenvalues on V are one. Call x semisimple or unipotent if $x=x_s$ or $x=x_u$. The first major result deals with the description of the conjugacy classes of semisimple elements.

An algebraic group is semisimple if it has no nontrivial normal, solvable algebraic subgroup, and it is simply connected if it has no nontrivial central extensions with finite kernel. Suppose G is such a group. There is a special distinguished set $V_1, \cdots, V_n$ of finite dimensional vector spaces with G-action (in technical language, the V_i are the representations corresponding to the fundamental weights of G for some choice of root system). Consider the functions X_i, $i=1, 2, \cdots, n$, on G where $X_i(x)$ is the trace of the matrix of the action of x on V_i, and consider the map $p: G\to k^{(n)}$ defined by $p(x)=(X_1(x), \cdots, X_n(x))$. Then the result is that p induces a bijection between the set of conjugacy classes of semisimple elements of G and $k^{(n)}$.

There is a vast amount of mathematics involved in establishing such a theorem. One needs to know that every semisimple element of G is contained in a maximal torus (an algebraic torus is a product of copies of the multiplicative group of the base field) and that all maximal tori are conjugate, both of which require deep results about Borel subgroups and Cartan subgroups. In addition, the fundamental theorem of representation theory of algebraic groups, which includes a knowledge of root systems, etc., is used in an important way. The above result on semisimple classes is only the first step in the analysis of conjugacy classes.

To go further, the following concept turns out to be useful: Call an element x of G regular if the conjugacy class of x is as large as possible in the sense of having the maximum algebraic dimension among all conjugacy classes. The regular semisimple elements are the ones belonging to a unique maximal torus. The existence of regular unipotent elements is more difficult; and the proof given in the notes requires a hypothesis on the characteristic of k. Now suppose further that G is semisimple and simply connected. Then the map p above actually induces a bijection between the set of conjugacy classes of regular elements of G and $k^{(n)}$, and moreover there is an algebraic cross-section to p which gives a representative of each regular class.

So far, then, the conjugacy classes of semisimple elements and the classes of regular unipotent elements have been determined. What remains, then, are the classes containing irregular unipotent elements. First, one remarks that the set V of unipotent elements of G is defined by polynomial conditions, hence is an algebraic subvariety of G, and then it is shown that the regular unipotents are precisely the nonsingular points of the variety V. Thus the irregular unipotents also form an algebraic subvariety W of G. Suppose, in addition to all our above assumptions, that G is simple. Then it turns out that W contains a unique dense (in the Zariski topology of W) conjugacy class C. A unipotent element x of G belongs to C if and only if the dimension of the centralizer of x in G is $r+2$, where r is the dimension of a maximal torus of G. Elements with this latter property are called subregular, and the last major results of the notes give some characterizations of subregular elements.

This brief summary of the notes' major results concerning the semisimple, regular, and subregular conjugacy classes should be regarded as just a hint of the wealth of information contained in the volume, and the types of arguments necessary to establish the main theorems.

The direct study of conjugacy classes just outlined is contained in the last half of the notes' 150 pages. The first half, in the author's words, "attempts an a priori development of the basic properties of affine algebraic groups . . . : the semisimple unipotent decomposition, conjugacy of Borel subgroups and of maximal tori, completeness of the variety of Borel subgroups, etc." As the author admits, there are "gaps in the actual development," usually accompanied by references to the literature, but by and large he has "attempted a coherent development."

By and large, he has been successful. Obviously any development which begins, as this one does, with the definition of affine algebraic variety and hopes to complete the classification theory of semisimple groups eighty pages later is going to have to be highly selective about what is actually proved. The reader is given a complete account of such things as the connected components of algebraic groups, the Jordan decomposition in algebraic groups, the triangularizability of unipotent matrix groups, and diagonalizable groups. This material should be accessible to readers without any prior knowledge of algebraic geometry or algebraic groups.

Next comes an account of quotients of affine algebraic groups by algebraic subgroups. Such quotients require the notion of abstract (nonaffine) varieties, and the reader without previous experience may find the two page introduction to such varieties in the notes somewhat sketchy—although even experienced readers will enjoy the compact account of complete varieties which follows. For actual existence of quotients, and a good deal of other material, the notes refer the reader to *Linear algebraic groups*, by Armand Borel (W. A. Benjamin, Inc., New York, 1969) for proofs.

The preliminary material concludes with a summary of the fundamental classification theorem for semisimple algebraic groups in terms of abstract root systems, and the fundamental theorem of representation theory of semisimple groups, which classifies representations in terms of dominant characters. While in both cases everything is carefully defined and stated, not much is proven or motivated, and beginners may find these sections difficult to appreciate.

In summary, Steinberg has mapped out an efficient route through algebraic groups from first principles to the deep theorems on conjugacy classes. Readers with some acquaintance with algebraic geometry and algebraic groups, who are prepared for an occasional ipsedixitism or feeling of serendipity, will find this a stimulating and rewarding volume. Novices who choose these notes as their first introduction to algebraic groups may find the going pretty tough at points and perhaps should take a look at Borel's book first.

These notes are taken from lectures the author gave at the Tata Institute of Fundamental Research during two months of 1972, and the volume reflects this lecture note style. There are a few misprints, typing errors, misspellings, omitted references, etc., all minor, and the sort of thing one expects in a volume of this type. Perhaps more important, the volume, like most lecture notes, seems to be more or less a transcription of what the lecturer wrote on the blackboard. One wishes that occasionally the insights and explanations most lecturers give orally as side comments would also make their way into lecture notes. In this volume, for example, we get a brief two-page introduction by the author at the beginning, and only rare hints in the body of the book as to what direction is being taken, why it works, and what the importance of the results are. Of course, the persevering reader will find this out in the end anyway, but some help from the author would have been useful.

ANDY R. MAGID