BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 1, January 1975

ULTRAFILTERS AND ALMOST DISJOINT SETS. II

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Communicated by W. Wistar Comfort, August 6, 1974

Unless otherwise stated, κ is an arbitrary infinite regular cardinal. For every infinite cardinal κ, $\mu\kappa$ is the family of uniform ultrafilters on κ. Our main result is:

THEOREM 1. *Suppose that* $2^\kappa = \kappa^+$. *Then for every* $U \in \mu\kappa$ *there is a family* $\{a_x\colon x \in U\}$ *such that: for every* $x \in U$, $a_x \subseteq x$ *and* $|a_x| = \kappa$; *and for every* $x, y \in U$ *with* $x \neq y$, $|a_x \cap a_y| < \kappa$.

This answers a question of Comfort communicated privately to the author and partially answers a question of Hindman [5]. It is still open whether Theorem 1 holds for singular κ as well. The hypothesis $2^\kappa = \kappa^+$ cannot be outright removed, since by [1] it is consistent with ZFC that there is no $A \subseteq P(\kappa)$ such that $|A| = 2^\kappa$, $|a| = \kappa$ for every $a \in A$, and $|a \cap b| < \kappa$ for every $a, b \in A$ with $a \neq b$. See [4], [5] and [8] for other relevant results.

DEFINITION 1. For $A \subseteq P(\kappa)$ and ideal $I \subseteq P(\kappa)$, I is said to be dense in A modulo sets of power $< \kappa$ if for each $x \in A$ such that $|x| = \kappa$, there is some $y \in I$ with $y \subseteq x$ and $|y| = \kappa$. For brevity we shall write "I is dense in A mod($< \kappa$)". I is dense mod($< \kappa$) if I is dense in $P(\kappa)$ mod($< \kappa$).

"I is λ-complete" is defined as in [7].

THEOREM 2. *For every* $U \in \mu\kappa$ *there is a* κ*-complete ideal* $I \subseteq P(\kappa) - U$ *such that* I *is dense* mod($<\kappa$).

Theorem 1 follows from Theorem 2 by induction on ordinals $< \kappa^+$, $a_x (x \in U)$ being chosen to belong to I. See [8, Theorem 1].

AMS (MOS) subject classifications (1970). Primary 02K25, 54D35.

Key words and phrases. Ultrafilter-uniform, regular, weakly selective, ideal-κ-complete, dense mod($< \kappa$), function-almost one-to-one, unbounded mod U.

[1] Research supported by NSF grant GP–43841.

Theorem 2 for $\kappa = \omega$ is trivial—let $I = P(\omega) - U$. Thus from now on, $\kappa > \omega$. Theorem 2 follows from Lemma 1 and Lemma 1 clearly follows from Lemmas 2 and 3.

LEMMA 1. *If $U \in \mu\kappa$, then there is a κ-complete ideal $I \subseteq P(\kappa) - U$ which is dense in U* mod$(< \kappa)$.

To prove Theorem 2, let I be as in Lemma 1 and set $x \in \bar{I}$ iff there are x_0, x_1 such that $x = x_0 \cup x_1$, $x_0 \in I$ and there is no $y \in I$ with $y \subseteq x_1$ and $|y| = \kappa$. Then $\bar{I}$ is as required.

For $f, g \in {}^\kappa\kappa$ we shall write $f \sim g$ (mod U) or $f < g$ (mod U) if $\{\rho\colon f(\rho) = g(\rho)\} \in U$ or $\{\rho\colon f(\rho) < g(\rho)\} \in U$ respectively.

DEFINITION 2 [2]. $U \in \mu\kappa$ is regular if there is a family $X = \{x_\xi\colon \xi \in \kappa\} \subset U$ such that for every infinite $S \subseteq \kappa$, $\bigcap\{x_\xi\colon \xi \in S\} = 0$.

DEFINITION 3 [6]. $f \in {}^\kappa\kappa$ is almost one-to-one if for each $\rho \in \kappa$, $|f^{-1}(\{\rho\})| < \kappa$.

DEFINITION 4. $f \in {}^\kappa\kappa$ is bounded mod U if for some $\alpha \in \kappa$, $\{\rho \in \kappa\colon f(\rho) < \alpha\} \in U$. Otherwise f is unbounded mod U.

DEFINITION 5. $U \in \mu\kappa$ is weakly selective if for every $f \in {}^\kappa\kappa$ which is unbounded mod U there is an almost one-to-one $g \in {}^\kappa\kappa$ such that $f \sim g$ (mod U).

LEMMA 2. *If $U \in \mu\kappa$ is either regular or not weakly selective, then there is a κ-complete $I \subseteq P(\kappa) - U$ which is dense in U* mod$(< \kappa)$.

PROOF. See [8, Theorems 5 and 11]. In outline, the proofs are as follows. Suppose first that U is regular. Let $X = \{x_\xi\colon \xi \in \kappa\}$ be as in Definition 2. Set

$$I = \{y \subseteq \kappa\colon \exists \eta < \kappa\ \forall \xi(\eta < \xi < \kappa \rightarrow |x_\xi \cap y| < \kappa)\}.$$

Then I is the desired ideal.

If U is not weakly selective, fix $f \in {}^\kappa\kappa$ unbounded mod U and such that there is no almost one-to-one g with $g \sim f$ (mod U). Set

$$I = \{y \subseteq \kappa\colon \forall \rho < \kappa(|y \cap f^{-1}(\{\rho\})| < \kappa)\}.$$

LEMMA 3. *If $U \in \mu\kappa$ is weakly selective and not regular, then there is a κ-complete ideal $I \subseteq P(\kappa) - U$ which is dense in U* mod$(< \kappa)$.

The next crucial lemma needed in the proof of Lemma 3 is due to A. Kanamori [6].

LEMMA 4 [6]. *If* $U \in \mu\kappa$ *is not regular then there is a least* (mod U) *almost one-to-one function* $f \in {}^{\kappa}\kappa$.

PROOF IN OUTLINE. Suppose that Lemma 4 is false. One can then construct by induction almost one-to-one functions $f_\alpha \in {}^{\kappa}\kappa$ $(\alpha \in \kappa)$ such that for all $\alpha < \beta < \kappa$, $f_\beta < f_\alpha$ (mod U), and in addition, if β is limit, then for all $\alpha < \beta$ and all $\rho \in \kappa, f_\beta(\rho) \leqslant f_\alpha(\rho)$. We now define sets $x_\alpha \in U$ $(\alpha < \kappa, \alpha$ successor) as follows. If $\alpha = \gamma + n$ where $\gamma < \kappa, \gamma$ is limit and $1 \leqslant n \in \omega$, then

$$x_\alpha = \{\rho < \kappa\colon \forall m \in \omega (0 \leqslant m < n \longrightarrow f_\alpha(\rho) < f_{\gamma+m}(\rho))\}.$$

It can be shown that $\{x_\alpha\colon \alpha < \kappa, \alpha \text{ successor}\}$ regularizes U.

LEMMA 5. *If* $U \in \mu\kappa$ *is weakly selective and not regular then there is a least* (mod U) $f \in {}^{\kappa}\kappa$ *unbounded* mod U.

PROOF. Immediate from Lemma 4.

LEMMA 6. (SCOTT, SEE [7, THEOREM 1.8]). *Let* $U \in \mu\kappa$ *and* $f \in {}^{\kappa}\kappa$ *be as in Lemma* 5. *Set* $V = \{x \subseteq \kappa\colon f^{-1}(x) \in U\}$. *Then* $V \in \mu\kappa$ *and the identity is a least* (mod V) *function unbounded* mod V. *Moreover* V *extends the filter of closed unbounded subsets of* κ.

SKETCH OF THE PROOF OF LEMMA 3. Let U, f and V be as in Lemma 6. Let J be the ideal of those $x \subseteq \kappa$ such that $\kappa - x$ contains a closed unbounded subset of κ. By Lemma 6, $J \subseteq P(\kappa) - V$. It is well known that J is κ-complete and dense mod$(< \kappa)$. Set

$$I = \{y \subseteq \kappa\colon \exists x \in J (y \subseteq f^{-1}(x))\}.$$

REFERENCES

1. J. E. Baumgartner, *Results and independence proofs in combinatorial set theory*, Doctoral Dissertation, Berkeley, Calif., 1970.

2. C. C. Chang and H. J. Keisler, *Model theory*, North-Holland, Amsterdam, 1973.

3. W. W. Comfort, private communication.

4. W. W. Comfort and N. Hindman, *Almost disjoint refining families for uniform ultrafilters*, Notices Amer. Math. Soc. 20 (1973), A–644.

5. N. Hindman, *On the existence of c-points in* $\beta N \backslash N$, Proc. Amer. Math. Soc. 21 (1969), 277–280. MR 39 #922.

6. A. Kanamori, *On irregularity* (a xeroxed note).

7. H. J. Keisler and A. Tarski, *From accessible to inacessible cardinals. Results holding for all accessible cardinal numbers and the problem of their extension to inaccessible ones,* Fund. Math. 53 (1963/64), 225–308. MR 29 #3385.

8. K. Prikry, *Ultrafilters and almost disjoint sets,* General Topology and Appl. 4 (1974), 269–282.

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