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AN ALGORITHM FOR THE TOPOLOGICAL DEGREE OF A MAPPING IN n-SPACE[1]

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1. **Introduction.** In this paper we announce a new formula for computing the topological degree $d(F, P, \theta)$, where $F = (f^1, \cdots, f^n)$ is a vector of real continuous functions mapping a polyhedron P in R^n into R^n, and θ is the zero vector in R^n.

Let $A = [a_{ij}]$ be an $n \times n$ real matrix, and let A_i denote the ith row of A. We use the convenient notation $\Delta_n(A_1, \cdots, A_n)$ for the determinant of A, and $|A_i| \equiv (a_{i1}^2 + \cdots + a_{in}^2)^{1/2}$ for the Euclidean norm of A_i.

Let $X_0, X_1, \cdots, X_q$ denote $q + 1$ points in R^n, where $q \leqslant n$, such that the vectors $X_i - X_0$, $i = 1, 2, \cdots, q$, are linearly independent. A q-simplex with vertices at $X_0, \cdots, X_q$ is defined by

$$S_q(X_0, \cdots, X_q) \equiv \left\{ X \in R^n: X = \sum_{i=0}^{q} \lambda_i X_i, \lambda_i \geqslant 0, \sum_{i=0}^{n} \lambda_i = 1 \right\}.$$

We denote by $[X_0 \cdots X_q]$ the oriented q-simplex, defined as in [2]. For example, if $q = n$, then $[X_0 \cdots X_q] = [X_0 \cdots X_n]$ is said to be positively (negatively) oriented in R^n if $\Delta_{n+1}(Z_0, \cdots, Z_n) > 0$ (< 0), where $Z_i = (1, X_i)$.

Let P be a connected, n-dimensional closed polyhedron represented as a "sum" of m' positively oriented n-simplexes in the form

$$P = \sum_{j=1}^{m'} [X_0^{(j)} \cdots X_n^{(j)}] \tag{1.1}$$

such that the intersection of any two of the simplexes has zero n-dimensional volume.

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The boundary of $[X_0 \cdots X_n]$ is represented in terms of oriented $n-1$-simplexes by

$$b[X_0 \cdots X_n] = \sum_{i=0}^{n} (-1)^i [X_0 \cdots X_{i-1} X_{i+1} \cdots X_n]$$

(see [2]). By means of this expansion, the boundary of P may be represented in the form

$$b(P) = \sum_{j=1}^{m} t_j [Y_1^{(j)} \cdots Y_n^{(j)}] \tag{1.2}$$

where P is defined in (1.1), and $t_j = \pm 1$. For example, if $n = 1$,

$$\begin{aligned} P &= [X_0X_1] + [X_1X_2] + \cdots + [X_{m-1}X_m], \\ b(P) &= [X_m] - [X_0]. \end{aligned} \tag{1.3}$$

Let F be a vector of n real C^1 functions defined on P, such that $F \neq \theta = (0, \cdots, 0)$ on $b(P)$. We denote by $d(F, P, \theta)$ the topological degree of F at θ relative to P. We define $d(F, P, \theta)$ by

$$\begin{aligned} d(F, P, \theta) &= \frac{1}{2}\left\{\frac{F(X_m)}{|F(X_m)|} - \frac{F(X_0)}{|F(X_0)|}\right\} \quad \text{if } n = 1, \\ d(F, P, \theta) &= \frac{1}{\Omega_{n-1}} \int_{b(P)} \frac{1}{|F|^n} \Delta_n\left(F, \frac{\partial F}{\partial u^1}, \cdots, \frac{\partial F}{\partial u^{n-1}}\right) du^1 \cdots du^{n-1} \\ &\qquad \text{if } n > 1, \end{aligned} \tag{1.4}$$

where Ω_{n-1} denotes the $n-1$ dimensional volume of the surface of the n-sphere, and where $F = F(X(U))$ is suitably parametrized as a function of $U = (u^1, \cdots, u^{n-1})$ (see [1, pp. 465–467]). If F is merely real and continuous on P, but not necessarily of class C^1, we define $d(F, P, \theta)$ by $d(F, P, \theta) = \lim_{(\nu \to \infty)} d(F^{(\nu)}, P, \theta)$, where $F^{(\nu)}$ is real and of class C^1 on P for $\nu = 1, 2, \cdots$, $\max_{(X \in P)} |F(X) - F^{(\nu)}(X)| \to 0$ as $\nu \to \infty$, and $d(F^{(\nu)}, P, \theta)$ is defined by means of (1.4).

The integral formula (1.4) is due to Kronecker [1, pp. 465–467]. Another integral for $d(F, P, \theta)$ has been given by Heinz [3]. In the following section we shall describe another procedure for evaluating $d(F, P, \theta)$, which depends only on the sign of the components of F at a finite number of points of $b(P)$.

2. **Formula for** $d(F, P, \theta)$. If a is a real number, we define $\operatorname{sgn} a$ by $\operatorname{sgn} a = -1, 0$ or 1 if $a < 0, = 0$ or > 0 respectively. We define $\operatorname{sgn} F$ by $\operatorname{sgn} F = (\operatorname{sgn} f^1, \cdots, \operatorname{sgn} f^n)$. Let us set

$$(2.1) \quad \delta_m(F, P, \theta) = \frac{1}{2^n n!} \sum_{j=1}^{m} t_j \Delta_n(\operatorname{sgn} F(Y_1^{(j)}), \cdots, \operatorname{sgn} F(Y_n^{(j)}))$$

where the t_j and $Y_i^{(j)}$ are the same as in (1.2). This formula is used to compute $d(F, P, \theta)$ by means of the following

ALGORITHM 2.1. (1) *Let* p *be a fixed positive integer.*

(2) *Set* $\delta = \delta_m(F, P, \theta)$ *as defined in* (2.1).

(3) *Revise the definition of* $b(P)$ *as follows: For* $j = 1, 2, \cdots, m$,

(a) *locate the longest segment* $\overline{Y_k^{(j)} Y_l^{(j)}}$ $(k < l)$ *of the oriented simplex* $t_j[Y^{(j)} \cdots Y_n^{(j)}]$ *in* (1.2), *and set* $A = (Y_k^{(j)} + Y_l^{(j)})/2$;

(b) *replace* $t_j[Y_1^{(j)} \cdots Y_n^{(j)}]$ *according to*:

$$(2.2) \quad \begin{aligned} &t_j[Y_1^{(j)} \cdots Y_k^{(j)} \cdots Y_l^{(j)} \cdots Y_n^{(j)}] \longleftarrow t_j[Y_1^{(j)} \cdots A \cdots Y_l^{(j)} \cdots Y_n^{(j)}], \\ &t_{j+m}[Y_1^{(j+m)} \cdots Y_k^{(j+m)} \cdots Y_l^{(j+m)} \cdots Y_n^{(j+m)}] \\ &\qquad\qquad \longleftarrow t_j[Y_1^{(j)} \cdots Y_k^{(j)} \cdots A \cdots Y_n^{(j)}]; \end{aligned}$$

(4) *replace* m *by* $2m$ *to get a new decomposition of* $b(P)$ *in terms of (twice as many) oriented simplexes.*

(5) *Set* $e = \delta_m(F, P, \theta)$ *as defined in* (2.1), *with the new* $b(P)$.

(6) *If* $\delta = e =$ *integer, go to* Step 6. *Otherwise set* $\delta = e$ *and return to* Step 3.

(7) *Replace* p *by* $p - 1$. *If the resulting* p *is positive, return to* Step 3. *Otherwise print out* m, δ.

Let us now make the following

ASSUMPTION 2.2. *Let* F *be continuous and real on* P, *where* P *is defined as in equation* (1.1). *Let* $b(P)$ *be defined as in equation* (1.2), *and let* $F \neq \theta$ *on* $b(P)$. *If* $n > 1$, *for all* $1 < \mu \leqslant n$, $\varphi^i = f^{j_i}$, $j_k \neq j_l$ *if* $k \neq l$, *and* $\Phi_\mu = (\varphi^1, \cdots, \varphi^\mu)$, *we assume that the sets* $T(A_\mu) = \{X \in b(P)\colon \Phi_\mu(X)/|\Phi_\mu(X)| = a_\mu\} \cap S_{\mu-1}$ *and* $b(P) - T(A_\mu)$ *consist of a finite number of connected subsets of* $b(P)$, *for all vectors* $a_\mu = (\pm 1, 0, \cdots, 0), (0, \pm 1, 0, \cdots, 0), \cdots, (0, \cdots, 0, \pm 1)$, *and for all* $\mu - 1$*-simplexes* $S_{\mu-1}$ *on* $b(P)$.

THEOREM 2.3. *If Assumption* 2.2 *is satisfied and if the integer* p *in Algorithm* 2.2 *is chosen sufficiently large, then Algorithm* 2.1 *prints out* *finite integers* m *and* δ, *where* $\delta = d(F, P, \theta)$, *and where* P *is defined in* (1.1).

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