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WEAK SEQUENTIAL CONVERGENCE IN THE DUAL OF A BANACH SPACE DOES NOT IMPLY NORM CONVERGENCE

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In this note we shall outline a proof of the following result (details will appear elsewhere).

For every infinite-dimensional Banach space E there is a sequence in E', the dual space, which tends to 0 in the weak topology $\sigma(E', E)$ but not in the norm topology. This is well known for separable or reflexive Banach spaces. See also [3] for other examples. The theorem has one of its main applications in the theory of holomorphic functions on infinite-dimensional topological vector spaces [6].

Let $L(E, F)$ denote the set of all bounded linear mappings from E into a Banach space F. Let l^∞ be the Banach space of all bounded sequences $z = (z_j)_{j=1}^\infty$ and C_0 the Banach space $C_0 = \{z \in l^\infty\colon z_j \to 0 \text{ as } j \to \infty\}$.

THEOREM. *There are* $\varphi_j \in E'$, $j \in \mathbf{N}$, *such that* $\|\varphi_j\| = 1$ *and* $\lim_{j\to\infty} \varphi_j(z) = 0$ *for all* $z \in E$.

PROOF. Let $F \subset E$ be a separable, infinite-dimensional subspace. From [1] and [2] it follows that there are $z^{(j)} \in F$ and $\psi_j \in E'$ such that $\|\psi_j\| = 1$, $\|z^{(j)}\| = 1$, $\psi_j(z^{(j)}) = 1$ and $\lim_{j\to\infty} \psi_j(z) = 0$ for every $z \in F$. Let $\psi \in L(E, l^\infty)$ be the mapping $\psi(z) = (\psi_1(z), \psi_2(z), \cdots, \psi_j(z), \cdots)$. Put $D = \psi(\bar{B}_E)$, where $\bar{B}_E$ is the closed unit ball in E. $\mathrm{Proj}_{[U]}\, D \cap \complement B_0$ is not compact for any infinite set $U \subset \mathbf{N}$, where B_0 is the open unit ball in C_0, $\complement$ denotes the complement and $\mathrm{Proj}_{[U]}(z_j)_{j=1}^\infty = (z_j')_{j=1}^\infty$, where $z_j = z_j'$ if $j \in U$ and $z_j' = 0$ if $j \notin U$. Put $N_U(z) = \overline{\lim}_{j,k\to\infty;j,k\in U} |z_j - z_k|$ for $z \in l^\infty$.

We shall say that E has *property* A if there are linear functionals as in the theorem.

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LEMMA 1. *If* E *does not have property* A *there exist an infinite set* $V \subset \mathbf{N}$ *and a number* $\epsilon > 0$ *such that* $\sup_{z \in D} N_U(z) > \epsilon$ *for every infinite* $U \subset V$.

The lemma follows easily from the fact that $\mathrm{Proj}_{[U]}\ D$ is not compact, hence, if E does not have property A, $\mathrm{Proj}_{[U]}\ D$ is not separable for any infinite $U \subset \mathbf{N}$.

LEMMA 2. *If there exist* $\varphi_n \in L(l^\infty, \mathbf{C}^n)$ *and* $C_k > 0$ *such that* $\sup_{z \in D} |\mathrm{Proj}_{[t]}\ \varphi_n(z)| \geqslant 1$ *for every* $n \in \mathbf{N}$ *and* $t \in \{1, 2, \cdots, n\}$, *and such that for every* $z \in D$, $n \in \mathbf{N}$ *and* $k \in \mathbf{N}$, $|\mathrm{Proj}_{[s]}\ \varphi_n(z)| \geqslant 2^{-k}$ *for at most* C_k *different* $s \in \{1, 2, \cdots, n\}$, *then* E *has property* A.

The lemma follows essentially from the fact that there are uncountably many $g_\alpha \in U_1 \times U_2 \times \cdots \times U_n \times \cdots$, where $U_n = \{1, 2, \cdots, n\}$, such that $\mathrm{Proj}_{[n]}\ g_{\alpha_1} = \mathrm{Proj}_{[n]}\ g_{\alpha_2}$ for at most finitely many $n \in \mathbf{N}$ if $\alpha_1 \neq \alpha_2$.
We shall recall the following

THEOREM (ROSENTHAL [7]). *There exists a surjection* $\varphi \in L(l^\infty, l^2(B))$, *where* $\mathrm{card}\ B = 2^{\mathrm{card}\,\mathbf{N}}$ *and* $l^2(B)$ *is the Hilbert space on* B.

We shall also recall the fact that if $\varphi' \in L(F, l^\infty)$, where $F \subset G$ is a subspace of a Banach space G, φ' can be extended to $\varphi'' \in L(G, l^\infty)$ such that $\|\varphi''\| = \|\varphi'\|$, by the Hahn-Banach theorem.

From this fact, the theorem of Rosenthal, and Lemmas 1 and 2, it is possible to prove

LEMMA 3. *Assume* E *does not have property* A. *Then there exist* $\varphi_n \in L(l^\infty, l^2(B))$, $H_n \subset B$, $z^{(n)} \in D$ *and* $X_n > 0$ *such that* $B \backslash H_n$ *is finite,* $H_n \subset H_{n-1} \subset \cdots \subset H_0 = B$, $\sup_{z \in D} \|\mathrm{Proj}_{[H_{k-1}]}\ \varphi_n(z)\| \leqslant X_k$ *if* $k \leqslant n$, $\|\mathrm{Proj}_{[H_{k-1} \backslash H_k]}\ \varphi_n(z^{(k)})\| > X_k/100$ *if* $k \leqslant n$ *and* $(X_k)_{k=1}^\infty$ *is not dominated by a convergent geometric series.*

To prove the theorem we suppose that E does not have property A. Then the sequence (X_n) in Lemma 3 may be taken to be decreasing; hence there exists, to every $k \in \mathbf{N}$, a number $n_k \in \mathbf{N}$ such that $X_{n_k}/X_{n_k+k} < 1 + 1/k$. But then

$$\left(\frac{100}{X_{n_k+k}}\ \mathrm{Proj}_{[H_{n_k-1} H_{n_k+k-1}]}\ \varphi_{n_k+k}\right)_{k=1}^\infty$$

and $C_n = 2^{2n} \cdot 200$ have the same properties as $(\varphi_k)_{k=1}^\infty$ and C_n in

Lemma 2, which is a contradiction. Q.E.D.

Let $H(F)$ be the set of holomorphic functions, that is to say locally bounded and analytic in the sense of Gâteaux, on a complex locally convex space F. A set $B \subset F$ is said to be *bounding* if $\sup_{z \in B} |f(z)| < \infty$ for every $f \in H(F)$.

COROLLARY 1. *If F is an infinite-dimensional, Hausdorff complex locally convex space, then every bounding set has an empty interior.*

PROOF. The Corollary follows from the theorem if we argue as in [2].

The results in this paper were announced in May 1973 at an international conference on infinite-dimensional holomorphy in Lexington, Kentucky.

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