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A CONSTRUCTIVE CHARACTERIZATION OF DISCONJUGACY

BY ANTON ZETTL[1]

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An ordinary linear differential operator L defined by

$$Ly = y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y \tag{1}$$

is said to be disconjugate on an interval I if every nontrivial solution of

$$Ly = 0 \tag{2}$$

has less than n zeros on I, multiple zeros being counted according to their multiplicity.

We assume $a_i \in C(I)$ for $i = 0, \cdots, n-1$. This assumption is made mainly for convenience and can be considerably weakened.

We announce here an algorithm for the construction of disconjugate operators of type (1) for any $n \geqslant 2$ and all intervals I. Our construction yields all disconjugate operators of type (1) if the interval I is either open or compact. This construction has the following features: It is iterative or inductive in the sense that the set of nth order disconjugate operators is constructed from the set of $(n-1)$st order ones. (The second order from the first order ones. All first order operators $y' + a_0(t)y$ are disconjugate.) The procedure for going from $n-1$ to n involves a parameter function.

For the remainder of this paper I denotes any compact or open interval, $C(I)$, the set of real valued continuous functions on I and $C'(I)$ the set of real valued functions on I which have continuous first derivatives.

THEOREM 1. *Given a_0 in $C(I)$ there exist $a_1, \cdots, a_{n-1}$ in $C(I)$ such that* (1) *is disconjugate. Moreover* (1) *is disconjugate if and only if there exists r in $C(I)$ and $b_0, \cdots, b_{n-2}$ in $C'(I)$ such that*:

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(i) $y^{(n-1)} + b_{n-2}y^{(n-2)} + \cdots + b_0 y$ *is disconjugate and*

(ii)
$$b_0' + rb_0 = a_0, \quad b_i' + rb_i + b_{i-1} = a_i \quad \text{for } i = 1, \cdots, n-2,$$
$$b_{n-2} + r = a_{n-1}.$$

From Theorem 1 the following algorithm for the construction of nth order disconjugate operators of type (1) is obtained: Start with a_0 in $C(I)$. Choose any r in $C(I)$. Let b_0 be any solution of $x' + rx = a_0$ i.e.,

$$b_0(t) = \exp\left(-\int_{t_0}^t r(s)ds\right)\left[c + \int_{t_0}^t \exp\left(\int_{t_0}^u r(s)ds\right) a_0(u)du\right]$$

for any t_0 in I and constant c. Choose $b_1, \cdots, b_{n-2}$ such that (i) holds. Determine a_i for $i = 1, \cdots, n-1$ by the second and third equations under (ii). Then the operator L determined by (1) is disconjugate. Furthermore all disconjugate operators L of type (1) on open or compact intervals are obtained this way.

For purposes of illustration we discuss this construction for the cases $n = 2$ and $n = 3$. The case $n = 2$: $Ly = y'' + a_1 y' + a_0 y$. The idea is to start with a given a_0 in $C(I)$ and characterize all functions a_1 for which L is disconjugate. The characterization is

$$a_1(t) = r(t) + \exp\left(-\int_{t_0}^t r(s)ds\right)\left[c + \int_{t_0}^t \exp\left(\int_{t_0}^u r(s)ds\right) a_0(u)du\right] \tag{3}$$

where t_0 is any point in I, c is an arbitrary constant, and r is an arbitrary function in $C(I)$.

In the literature, disconjugacy of second order equations is usually discussed for equations in the form $y'' + qy$ or $(ry')' + qy$. Consider $y'' + qy$. The question of disconjugacy for this equation reduces to: When is a_1 in (3) (with $a_0 = q$) zero? Setting $a_1(t) \equiv 0$ and differentiating reduces (3) to $r' + r^2 + a_0 \equiv 0$. This is the famous Riccati equation associated with the form $y'' + a_0 y$. So our characterization (3)—in the second order case for the special form $y'' + a_0 y$—reduces to the well-known equivalence between disconjugacy and existence of solutions of the Riccati equation.

The case $n = 3$: $Ly = y''' + a_2 y'' + a_1 y' + a_0 y$. Again the idea is to start with any function a_0 and determine all pairs of functions (a_1, a_2) which make L disconjugate. This is done as follows: Let r be in $C(I)$ and let b_0 be any solution of $x' + rx = a_0$. Take any b_1 for which

$y'' + b_1 y + b_0 y$ is disconjugate (i.e., determine b_1 as discussed in the case $n = 2$ above). Now let $a_1 = b_1' + b_0 + rb_1$ and $a_2 = r + b_1$. Then L is disconjugate and all third order disconjugate operators L of type (1) on open or compact intervals are obtained this way.

The proof of Theorem 1 is based on the following idea: The nth order operators (1) which are disconjugate are precisely those determined by products

$$(y' + ry)(y^{(n-1)} + b_{n-2}y^{(n-2)} + \cdots + b_0 y)$$

where the $(n-1)$st order operator with coefficients b_i is disconjugate. Product here is meant in the sense of composition. A detailed proof and related matters will be published elsewhere.

DEPARTMENT OF MATHEMATICAL SCIENCES, NORTHERN ILLINOIS UNIVERSITY, DEKALB, ILLINOIS 60115

Current address: Department of Mathematics, The University, Dundee, Scotland