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RELATIVE COMPLETIONS OF A-SEGAL ALGEBRAS

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We announce some new results about multipliers and ideal theory of A-Segal algebras and their relative completions. Complete details are to appear elsewhere [3], [4]. The results about multipliers (Theorem 6) represent work done jointly with Richard R. Goldberg [4].

DEFINITIONS. If A is a Banach algebra, we say the subalgebra $B \subseteq A$ is an A-Segal algebra provided B is a dense left ideal of A, B is a Banach algebra with respect to a norm $\| \ \|_B$, the injection of B into A is continuous, and multiplication is (jointly) continuous on $A \times B$ into B. We shall *always* suppose that A does not have an identity.

The relative completion of B with respect to A, denoted $\widetilde{B}^A$, is defined by

$$\widetilde{B}^A = \bigcup_{n>0} \overline{S_B(\eta)}^A,$$

where $S_B(\eta) = \{f \in B \mid \|f\|_B \leqslant \eta\}$ and $\overline{E}^A$ is the A closure of E. For $f \in \widetilde{B}^A$ we define $|||f|||$ by

$$|||f||| = \inf\{\delta \mid f \in \overline{S_B(\delta)}^A\}.$$

THEOREM 1. *If B is an A-Segal algebra, then $\widetilde{B}^A$ (with norm $||| \ |||$) is an A-Segal algebra. Furthermore, if B has right approximate units which are bounded in the A-norm, then B is a closed left ideal of $\widetilde{B}^A$ and the embedding of B into $\widetilde{B}^A$ is isometric if the approximate units have A-norm one.*

In case A and B share common right approximate units of A-norm one, then $\widetilde{B}^A$ has a rather simple description which permits a straightforward proof of all of the assertions of Theorem 1. Such is the case when $A = L^1(G)$ and B is an ordinary Segal algebra [6, p. 16]. Indeed,

THEOREM 2. *With A and B as in the preceding paragraph and U denoting (a set of) common right approximate units, we have*

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$$f \in \widetilde{B}^A \Leftrightarrow M \equiv \operatorname{Sup}\{\|u * f\|_B \mid u \in U\} < \infty,$$

and in this case $|||f||| = M$.

From here on we suppose that A and B have common right approximate units of A-norm one. The following theorem, which has the assertion of the second sentence in Theorem 1 as a consequence, is of independent interest.

THEOREM 3. $S_B(\delta) = \overline{S_B(\delta)}^A \cap B$; *in particular, if* $B = \widetilde{B}^A$, *then* $S_B(\delta) = \overline{S_B(\delta)}^A$.

DEFINITION 4. We say B is singular provided $B \neq \widetilde{B}^A$.

Perhaps the simplest example of a singular A-Segal algebra and its relative completion is the pair $(C(G), L^\infty(G))$, where G is an infinite compact group and $A = L^1$. Additional examples of singular Segal algebras are given in [**3**] and [**4**]; a more detailed analysis of singularity is given in [**3**].

Some results which are useful for an analysis of multipliers and the ideal theory of A-Segal algebras and their relative completions are summarized in

THEOREM 5. (1) *If* B *is a closed ideal in the* A-*Segal algebra* B_1, *then* $B_1 \subseteq \widetilde{B}^A$. *Let* U *denote right approximate units for* B. (2) *If* $f \in \widetilde{B}^A$, *then* $f \in B \Leftrightarrow$ *given any* $\epsilon > 0$ *there exists* $u(f, \epsilon) \equiv u \in U$ *so that* $|||uf - f||| \leqslant \epsilon$. (3) $A\widetilde{B}^A \subseteq B$ *and, hence,* $\widetilde{B}^A \cdot \widetilde{B}^A \subseteq B$. *We thus see that* $\widetilde{B}^A$ *fails to factor if* B *is singular.*

We now specialize to the case where $A = L^1(G)$, and $B = S(G)$ is a symmetric Segal algebra as defined by H. Reiter [**6**, p. 17]. Here, G denotes a locally compact nondiscrete group. The (multiplier) algebra of bounded linear operators from $L^1(G)$ into $S(G)$ $(\widetilde{S}^{L^1}(G))$ for which $T(f * g) = f * Tg$ is denoted (L^1, S) $((L^1, \widetilde{S}^{L^1}))$.

THEOREM 6. *Let* $\langle e_\alpha \rangle$ *be a bounded approximate identity for* $L^1(G)$. *For a measure* $\mu \in M(G)$ *the following three conditions are equivalent*:
(1) $\operatorname{Sup}_\alpha \|e_\alpha * \mu\|_S < \infty$; (2) $\mu \in (L^1, S)$; (3) $\mu \in (L^1, \widetilde{S}^{L^1})$.

Furthermore, if $(L^1, S) \subseteq L^1(G)$, *then* (L^1, S) *is isometrically isomorphic with* $\widetilde{S}^{L^1}$.

For our final theorems we require that G be an infinite compact group. All unexplained notation may be identified from the analogous results in [**5**].

THEOREM 7 [**5**, 38.9, p. 453]. *Let* $S(G)$ *be a singular Segal algebra. Let*

P be any subset of Σ. Let $\mathcal{F}$ be a closed linear subspace of $\widetilde{S}^{L^1}(G)$ for which $\mathcal{F} \cap S(G) = S_p(G)$ and $\mathcal{F} \subset \widetilde{S}_p^{L^1}(G)$. Then $\mathcal{F}$ is a closed two-sided ideal in $\widetilde{S}^{L^1}(G)$; conversely, all closed two-sided ideals in $\widetilde{S}^{L^1}(G)$ have this form. Furthermore, the quotient algebra $\widetilde{S}^{L^1}(G)/S(G)$ is a zero algebra. The closed two-sided ideals in $\widetilde{S}^{L^1}(G)$ for which the quotient algebra is a zero algebra are exactly the closed linear subspaces of $\widetilde{S}^{L^1}(G)$ that contain $S(G)$.

THEOREM 8. *Suppose $S(G)$ is a singular Segal algebra. For each $\sigma \in \Sigma$, $\widetilde{S}^{L^1}_{\{\sigma\}}(G)$ is a regular maximal proper two-sided ideal in $\widetilde{S}^{L^1}(G)$. If M is a nonzero bounded linear functional on $\widetilde{S}^{L^1}(G)$ which vanishes on $S(G)$, then $M^{-1}(0)$ is a closed maximal proper two-sided ideal in $\widetilde{S}^{L^1}(G)$ for which $\widetilde{S}^{L^1}_{\{\sigma\}}(G)/M^{-1}(0)$ is a 1-dimensional zero algebra. Every maximal closed proper two-sided ideal of $\widetilde{S}^{L^1}(G)$ not of the form $\widetilde{S}^{L^1}_{\{\sigma\}}(G)$ is obtained in this way.*

For the ideal theory of A-Segal algebras with approximate identities, we refer to [1] and [2].

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