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THE CONJUGACY PROBLEM AND CYCLIC AMALGAMATIONS

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Max Dehn first posed the word and conjugacy problems for groups, and solved these problems [2] for the fundamental group G_k for an orientable 2-manifold of genus k. This group has the presentation

$$G_k = (a_1, b_1, \cdots, a_k, b_k; a_1^{-1}b_1^{-1}a_1b_1 \cdots a_k^{-1}b_k^{-1}a_kb_k = 1).$$

We note that G_k is a free product of two free groups with a cyclic amalgamation generated by nonpowers.

The author generalized Dehn's result [3] by solving the conjugacy problem for any free product of free groups with a cyclic amalgamation. On the other hand, Miller [5] gave an example of a free product of two free groups amalgamating finitely generated subgroups which has an unsolvable conjugacy problem. Thus a cyclic amalgamation seems an essential criteria in finding classes of groups with solvable conjugacy problems. (For notational convenience we will speak of a free product "amalgamating u and v" when we mean "amalgamating the cyclic subgroups generated by u and v".)

Anshel and Stebe solved the conjugacy problem [1] for certain *HNN* extensions where the underlying group is free and the extension is obtained by an isomorphism of cyclic subgroups. Following Anshel and Stebe, we say that an element h in a group G is *non-self-conjugate* if its distinct powers are in different conjugacy classes. We will also say that h is *power-solvable* if for any w in G we can decide whether or not w is a power of h. (A group has a *solvable power problem* if all its elements are power-solvable.) We note that every nonidentity element in a free group is non-self-conjugate and power-solvable.

We now are able to state our main result which clearly generalizes Dehn's result.

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THEOREM 1. *Let u be any nonpower in a free group A. Let v be a non-self-conjugate and power-solvable element in a group B with a solvable conjugacy problem. Then the free product of A and B amalgamating u and v has a solvable conjugacy problem.*

The requirement that u be a nonpower in a free group A is actually stronger than needed. In order to concisely restate Theorem 1 in a more general form, we introduce a definition. An element h in a group G will be called a *critical* element if it has the following four properties:

(a) h is non-self-conjugate.

(b) h is *conjugate-power-solvable*, i.e. for any w in G we can decide whether or not w is conjugate to a power of h.

(c) h is *double-coset-solvable*, i.e. for any pair u, v in G we can decide whether or not there exist integers r and s such that $h^r u h^s = v$.

(d) If $h^m u = u h^m$, then u is a power of h.

We will also say that h is *semicritical* if h satisfies the first three of the four properties.

The author's main technical result in [3] shows that elements in free groups are double-coset-solvable. Such elements are clearly conjugate-power-solvable. Nonpowers also satisfy property (d). Hence nonidentity elements in a free group are semicritical, and nonpowers are critical. Accordingly, the following two theorems generalize Theorem 1 and the main result in [3].

THEOREM 2. *Let A and B be groups with solvable conjugacy problems. Let u be a critical element of A and let v be a non-self-conjugate and power-solvable element of B. Then the free product of A and B amalgamating u and v has a solvable conjugacy problem.*

THEOREM 3. *Let G be the free product of groups with solvable conjugacy problem amalgamating a cyclic subgroup generated by semicritical elements in the factors. Then G has a solvable conjugacy problem.*

All relevent terms appear in the text by Magnus, Karrass and Solitar [4]. Details and proofs of the above results will appear elsewhere.

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