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ON WEAK AND STRONG CONVERGENCE OF POSITIVE CONTRACTIONS IN L_p SPACES

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We consider a linear operator T mapping an L_p space into itself. T will be assumed to be *positive* $(f \geqslant 0 \Rightarrow Tf \geqslant 0)$ and a contraction $(\|T\| \leqslant 1)$. A matrix (a_{ni}), $n, i = 1, 2, \cdots$, is called *uniformly regular* if

$$\sup_n \sum_i |a_{ni}| < \infty, \quad \lim_n \sup_i |a_{ni}| = 0, \quad \lim_n \sum_i a_{ni} = 1. \tag{1}$$

THEOREM 1. *If* $1 < p < \infty$ *and if* T *is a positive contraction on* L_p, *then the following conditions are equivalent*

(A) $\lim_n T^n$ *exists in the weak operator topology,*

(B) $\lim_n \Sigma_i a_{ni} T^i$ *exists in the strong operator topology for every uniformly regular matrix* (a_{ni}).

The theorem is already known for $p = 1$ and for $p = 2$, even for not necessarily *positive* contractions ([2], [5]).

SKETCH OF PROOF. (i) The implication (B) $\Rightarrow$ (A) is easy and holds in more general situations (cf. [5]). Hence we only prove (A) $\Rightarrow$ (B).

(ii) If G is the largest set such that G supports a T-invariant function g, then $f \in L_p(G)$ implies that $Tf \in L_p(G)$. $Tg = g$ implies $T^* g^{p-1} = g^{p-1}$, hence $f \in L_{p/(p-1)}(G)$ implies $T^* f \in L_{p/(p-1)}(G)$. Therefore there is no communication between G and $F = G^c$, and the restrictions of T to $L_p(G)$ and $L_p(F)$ may be considered separately. On G there exists a strictly positive T-invariant function, and therefore the theorem for $L_p(G)$ follows from the results proved in [5, §2]. There exist no nontrivial positive T-invariant functions on F, and hence the weak limit of T^n restricted to $L_p(F)$ must be zero.

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(iii) Assuming that T^n converges weakly to zero, (A) $\Rightarrow$ (B) is proved for invertible isometries. The computation is made possible by the application of the elementary identity $(T^{-n}f)^{p-1} = T^*(f^{p-1})$. In fact, one obtains estimates uniform in the following sense: The rate of weak unaveraged convergence determines the rate of strong convergence of averages independently of the operator and the function.

(iv) The uniformity allows the approximation by finite dimensional operators for which the contraction case is reduced to the invertible isometry case by the application of a dilation theorem proved in [1]. For detailed proofs see [4].

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