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BOOK REVIEW

Topological Riesz spaces and measure theory, by D. H. Fremlin, Cambridge University Press, New York, 1974, xiv+266 pp., $17.50

A *Riesz space*, or *vector lattice*, is a partially ordered real vector space which, as a partially ordered set, is a lattice. Many of the most important spaces of functions occurring in analysis, and in particular in measure and integration theory, are Riesz spaces, and it is therefore not surprising that the abstract theory of Riesz spaces should play a significant role in the study of such function spaces. In fact, it is equally true that the Riesz space structure of function spaces has had a profound influence on the development of the abstract theory. The purpose of D. H. Fremlin's book is to identify those concepts in the abstract theory of Riesz spaces which have particular relevance to the older discipline of measure theory, and to show how the corresponding aspects of measure theory may be illuminated using the techniques of functional analysis on Riesz spaces.

The book is primarily addressed to functional analysts, and a basic knowledge of functional analysis, including the rudiments of the theory of topological vector spaces, is assumed. A knowledge of the basics of abstract measure theory will also be of considerable help to the reader, although strictly speaking this is formally unnecessary. Otherwise, no specialized knowledge is required and the book will consequently be accessible to a reasonably well-educated graduate student.

The contents fall loosely into four sections. Chapters 1–3 give an account of the basic theory, both algebraic and topological, of Riesz spaces. This is followed by three chapters in which the theory is put to work to give a development of abstract measure theory, after which comes a chapter on the representation by integrals of linear functionals on function spaces. The book concludes with a chapter on weakly compact sets in Riesz spaces.

The basic algebraic theory of Riesz spaces is given in Chapter 1. Firstly, the fundamental results connecting the algebraic and lattice structures are obtained, and further important concepts (e.g. Dedekind completeness, quotient spaces, Archimedean spaces) are introduced. The discussion then turns to the various classes of linear mappings between Riesz spaces, and it is here that the real force of the axioms of a Riesz space become apparent. The most important spaces of mappings associated with a Riesz space $\boldsymbol{E}$ are the two (order) dual spaces $\boldsymbol{E}^{\sim}$ and $\boldsymbol{E}^{\times}$. The space $\boldsymbol{E}^{\sim}$ consists of those linear functionals on $\boldsymbol{E}$ which are expressible as the difference of two increasing linear functionals, while $E^{\times}$ is the subspace of $E^{\sim}$ consisting of those functionals which are expressible as the difference of two order-continuous increasing linear functionals. (An increasing linear mapping between two Riesz spaces is order-continuous if it preserves suprema of upward-directed sets.) The spaces $E^{\sim}$ and $E^{\times}$ have obvious partial orders under which they are partially ordered vector spaces. What is not so obvious is the fact that they are then Riesz spaces, and are even Dedekind complete.

Chapter 2 is devoted to a study of topological Riesz spaces and to an examination of relationships between their order and topological structures. Much of the chapter is taken up in describing various topological properties which have special relevance to Riesz spaces, and in investigating how the topological dual E' of a topological Riesz space E is related to $E^{\sim}$ and $E^{\times}$.

The simplest requirement of a linear space topology on a Riesz space is that the positive cone be closed, such topologies being called *compatible*. Although this condition does have some simple consequences (for instance, it forces the Riesz space to be Archimedean), it is necessary to impose other topological conditions to obtain an interesting theory. Of the various types of topologies introduced, the most important are probably the *locally solid* topologies (0 has a base of neighbourhoods consisting of solid sets) and the *Lebesgue* topologies (topologically closed sets are order-closed). A sample result: if E is a Riesz space with a compatible Lebesgue topology, then $E' \subset E^{\times}$. The relationship between Dedekind (order) completeness and topological completeness is also investigated and Nakano's theorem is proved. The chapter concludes with a section on complete metrizable topologies on Riesz spaces, followed by a short discussion of L- and M-spaces.

Two general features of Chapter 2 are worthy of mention. Firstly, the hypothesis of local convexity is not, in general, assumed. This does not cause too much extra work and is a worthwhile level of generalization, since there are important examples of topological Riesz spaces (e.g. spaces of measurable functions endowed with the topology of convergence in measure) which are not locally convex but which are amenable to analysis using Riesz space techniques. Secondly, many of the ideas in Chapter 2 have previously been considered only in more restricted settings, and some have appeared before only in various research articles. Consequently, the unified treatment presented here is of considerable value.

In Chapter 3, a brief account is given of the duality between a Riesz space and its order dual spaces, the most important idea being the expected (but nontrivial) homomorphic representation of an Archimedean Riesz space E as a subspace of $E^{\times\times}$.

The principal aim of this book is to show how the abstract theory of Riesz spaces relates to measure theory and, having developed the necessary Riesz space theory, the author now turns to this task.

A concept which is often paid little or no attention in traditional accounts of abstract measure theory is that of the *measure algebra* of a measure space. This is defined as follows. Let (X, Σ, μ) be a measure space; that is, let μ be a countably additive measure on a σ-algebra Σ of subsets of a set X. Considering Σ as a Boolean algebra, the collection Σ_0 of μ-null sets is an ideal in Σ; and so the quotient Boolean algebra Σ/Σ_0 can be formed. Furthermore, μ factors through Σ_0 to define a function $\mu^{\cdot}$ on this quotient. The pair $(\Sigma/\Sigma_0, \mu^{\cdot})$ is the measure algebra associated with (X, Σ, μ). The broad strategy of the approach to abstract measure theory developed in Chapters 4–6 is to construct certain Riesz spaces from this measure algebra and to show that these spaces can be identified with the usual L^1 and L^∞ spaces of the underlying measure space. The abstract theory developed in the earlier part of the book can then be applied to obtain much of the theory of measure and integration.

The Riesz spaces constructed from $(\Sigma/\Sigma_0, \mu^{\cdot})$ are easily described. For convenience, write $\mathfrak{A}$ for Σ/Σ_0 and let $\mathfrak{A}^f$ be the set of elements in $\mathfrak{A}$ of finite $\mu^{\cdot}$-measure. The Boolean ring $\mathfrak{A}$ has a canonical representation as a ring of subsets of its Stone space Y. Let φ be this representation and denote by $\boldsymbol{S}(\mathfrak{A})$ the linear span of the characteristic functions of the sets $\{\varphi(a): a \in \mathfrak{A}\}$. The space $\boldsymbol{L}^\infty(\mathfrak{A})$ is the uniform closure in $l^\infty(Y)$ of $\boldsymbol{S}(\mathfrak{A})$. The space $\boldsymbol{S}(\mathfrak{A}^f)$ is constructed in a similar way from $\mathfrak{A}^f$, and it is straightforward to show that $\mu^{\cdot}$ induces a norm on $\boldsymbol{S}(\mathfrak{A}^f)$ (this is essentially the same process as that of defining the L^1 seminorm on the space of integrable simple functions on a measure space). The space $\boldsymbol{L}^1(\mathfrak{A}, \mu^{\cdot})$ is the normed space completion of $\boldsymbol{S}(\mathfrak{A}^f)$ with respect to this norm. It is a simple matter to identify $\boldsymbol{L}^\infty(\mathfrak{A})$ as a Banach Riesz space (i.e. as a Banach lattice) with the usual space $L^\infty(\Sigma, \mu)$ and, with more difficulty, $\boldsymbol{L}^1(\mathfrak{A}, \mu^{\cdot})$ can be identified with $L^1(\Sigma, \mu)$.

Of particular importance in the discussion of the Riesz spaces associated with $(\mathfrak{A}, \mu^{\cdot})$ is the existence of a natural duality between $\boldsymbol{L}^\infty(\mathfrak{A})$ and $\boldsymbol{L}^1(\mathfrak{A}, \mu^{\cdot})$ which, when the underlying measure space is *semifinite* (given E with $\mu(E)=\infty$, there exists $F \subset E$ with $0<\mu(F)<\infty$), identifies $\boldsymbol{L}^1(\mathfrak{A}, \mu^{\cdot})$ with $\boldsymbol{L}^\infty(\mathfrak{A})^\times$. This fact is essentially an abstract form of the Radon-Nikodým theorem, the usual theorem following easily from it. The familiar representation of the topological dual of $L^1(\Sigma, \mu)$ as $L^\infty(\Sigma, \mu)$ also arises from this duality. Assuming that (Σ, μ) is semifinite, $L^\infty(\Sigma, \mu)$ 'equals' the Banach space dual of $L^1(\Sigma, \mu)$ precisely when the Boolean algebra $\mathfrak{A}$ is Dedekind complete (as a lattice). Measure spaces for which this is true are often called *localizable*, and of course include the σ-finite spaces.

The work involved in investigating the Riesz spaces associated with a measure algebra is considerable and should be thought of not as providing a 'soft' analysis alternative to a more traditional approach to measure theory, but rather as giving a new perspective to an established theory.

In Chapters 4–6, the methods of functional analysis are used to discuss aspects of abstract measure theory. On the other hand, one area of functional analysis to which measure theory has made important contributions concerns the representation by integrals of linear functionals on spaces of continuous functions. This topic is taken up in Chapter 7. The fundamental result here is, of course, the Riesz Representation Theorem, in which the underlying topological space is locally compact and the continuous functions involved vanish at infinity. However, more general situations have attracted attention in recent years, the motivation coming in part from probability theory, and the presentation here is directed towards some of these more recent developments. The approach is similar to that given by F. Topsøe, *Topology and measure*, Lecture Notes in Math., vol. 133, Springer, 1970. It should also be mentioned that the properties of Riesz spaces play a very minor role here, and that the discussion is almost entirely independent of the rest of the book. However, since the chapter provides an excellent account of an important area in which ideas from measure theory, functional analysis and topology blend together, it is most worthy of inclusion.

In the final chapter of the book, the author returns to the abstract theory of Riesz spaces and, more particularly, to a study of the weak (star) compact sets in their order duals. Since this chapter is more specialized and will be of principal interest to topological vector space experts, I will not discuss it in any detail.

The layout of the book has some welcome features. Each section begins with a useful summary advising the reader of the salient points to look out for, and ends with 'Notes and Comments', in which the material is put in perspective, references on which it is based are given, and suggestions for further reading and possible lines of research are made. At the end of each chapter (excluding Chapter 3) the author gives a variety of examples illustrating the theory, and there are numerous exercises in which ideas from the text are developed further. The exposition is lively, at times somewhat informal, but always clear and precise. This is a thought-provoking book which will be of interest to many analysts.

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