BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 1, January 1975

BOOK REVIEW

Polynomial approximation, by Robert P. Feinerman and Donald J. Newman, Williams & Wilkins, Baltimore, Maryland, 1974, viii+148 pp.

Where is the subject matter of the book under review to be found on the mathematical map? If we turn to Bourbaki's *Elements of mathematics* (English translation), *General topology*, Part 2, Chapter X, *Function spaces*, Historical Note, p. 348, we read that "$\cdots$ Weierstrass himself discovered the possibility of uniform approximation of a continuous real-valued function in one or more variables on a bounded set by polynomials. This result immediately aroused lively interest and led to many "quantitative" studies (*)." (The (*) refers to de La Vallée Poussin's, *Leçons sur l'approximation des fonctions d'une variable réele*, Paris (Gauthier-Villars), 1919.) This book is situated precisely among the "quantitative" studies. Of its twelve chapters eleven are devoted to quantitative aspects of the Weierstrass polynomial approximation theorem and all but one of these to functions of one variable. The anomalous chapter (Chapter IX) is devoted to functions of more than one variable.

Let us be a bit more precise. Let P_n be the set of (real) polynomials of degree at most n and let $\|\cdot\|$ be the uniform norm in $C[0, 1]$; then given $f\in C[0, 1]$ and an integer $n\geqq 0$, there exists $p^*\in P_n$ (called the best approximation to f out of P_n) such that $\|f-p^*\|=\inf[\|f-p\|:p\in P_n]=E_n(f)$. The Weierstrass theorem is the proposition that $E_n(f)\rightarrow 0$ as $n\rightarrow\infty$. The typical quantitative question which is a major concern of this book is: how fast does $E_n(f)$ go to zero? A key result of this type is due to Dunham Jackson. It says that if $f\in C[0, 1]$ then $E_n(f)\leqq c\omega(f, (1/n))$, where c is a constant (shown to be $\leqq 17$ in this book) and $\omega(f, \delta)$ is the modulus of continuity of f, i.e., $\omega(f, \delta)=\sup[|f(x)-f(y)|: |x-y|\leqq\delta]$. Moreover, this result is sharp in the sense that $\omega(f, (1/n))$ cannot be replaced by any other function of f and n which goes to zero faster, as $n\rightarrow\infty$, for all f. Indeed, as the authors show, if $S=\{f:|f(x)-f(y)|\leqq|x-y|\}$, there is an $f\in S$ such that $E_n(f)\geqq 1/(2n+2)$. We must mention that there is an analogous theory of best approximation in the uniform norm to continuous 2π-periodic functions by trigonometric polynomials of degree at most n with similar and sometimes equivalent results. The "periodic case" is also extensively studied in this book.

We turn now to a chapter by chapter survey of the highlights of the book. In the first chapter several familiar proofs of the Weierstrass theorem for functions of one variable are given. The second chapter provides some needed properties of polynomials. In Chapter III the existence and uniqueness of a best polynomial approximation out of P_n is demonstrated, as is the remarkable Chebyshev characterization of such a best approximation, namely, given $f \in C[0, 1]$, $p \in P_n$ is a best approximation to f ($\notin P_n$) if, and only if, there exist $n+2$ points $0 \leqq x_0 < x_1 < \cdots < x_{n+1} \leqq 1$ such that

$$|f(x_i) - p(x_i)| = \|f(x) - p(x)\|, \quad i = 0, \cdots, n+1,$$

and the numbers $[f(x_i) - p(x_i)]$ alternate in sign. The fourth chapter is devoted to Jackson-type theorems and the fifth to results in the periodic case due to Bernstein and Zygmund which are inverse to those of Jackson type, that is, which conclude that a function must be of a certain smoothness if the error of approximation goes to zero fast enough.

The operator which maps a function into its best approximation is, in general, nonlinear. In Chapter VI the effectiveness of linear operators in producing uniform approximations is considered. If $\{L_n\}$ is a sequence of bounded positive linear operators from $C[a, b]$ to P_n, then Bohman and Korovkin showed that $\|L_n f - f\| \to 0$ for all $f \in C[a, b]$ if, and only if $\|L_n x^i - x^i\| \to 0$ for $i = 0, 1, 2$. This striking result, which has inspired a large literature of its own, is given a quantitative proof following Shisha and Mond.

Until now most of the material has been quite well known. With Chapter VII, Rational Approximation, new material due to the authors begins to appear with increasing frequency. The issue in Chapter VII is rationals vs. polynomials. It turns out that for some classes of functions, such as S, rational functions are, in general, no better than polynomials as approximators. However, for the function $f(x) = |x|$ (which plays such an important role in this whole subject), which serves as an exemplar of certain classes of functions (piecewise analytic functions, etc.), rational approximation is a dramatic improvement over polynomial approximation. In Chapter IX the previous notions are generalized to approximating an element of a real normed linear space out of a finite dimensional subspace. The ninth chapter gives a Jackson theorem for certain sets in spaces of dimension more than one.

Chapter X deviates from the quantitative theme and is devoted to the completeness question, i.e., the Weierstrass kind of question. Here we find some theorems of Müntz type. For example: If $\frac{1}{2} < \lambda(n)$, then $\{1, x^{\lambda(n)}\}$ is complete in $C[0, 1]$ if, and only if $\Sigma(1/\lambda(n)) = \infty$. Also some new results like: $\{\cos nx\}$, $n = 1, 2, \cdots$, is incomplete in $L^1[0, \pi]$ but $\{\cos nx\}$, $n = 0, 1, \cdots$, is complete in $C[0, \pi]$. The eleventh chapter contains what the reviewer believes to be the most beautiful of the new material, namely, a Müntz-Jackson theorem. The setting is now the following. We are given $f \in C[0, 1]$ and a set $\Lambda = \{\lambda(0), \cdots, \lambda(n)\}$. How well can "polynomials", $\Sigma c_k x^{\lambda(k)}$, approximate f? Put

$$E_\Lambda(f) = \inf_{c_0,\cdots,c_n} \left[\left\| f - \sum_{i=0}^{n} c_i x^{\lambda(i)} \right\| \right].$$

The authors show that there is an "approximation index" $\varepsilon(\Lambda)$ such that $(\frac{1}{50})\omega(f, \varepsilon(\Lambda)) \leqq E_\Lambda(f) \leqq 368\omega(f, \varepsilon(\Lambda))$. (When $\lambda(i) = i$ we saw that $\varepsilon(\Lambda) = 1/n$.) The computation of $\varepsilon(\Lambda)$ poses difficulties, but if $0 = \lambda(0) < \lambda(1) < \cdots < \lambda(n)$ and $\lambda(k) - \lambda(k-1) \geqq 2$ (the "separated case"), the authors show that

$$\frac{1}{2} \exp\left(-2 \sum_{k=1}^{n} \frac{1}{\lambda(k)}\right) \leqq \varepsilon(\Lambda) = \prod_{i=1}^{n} \left| \frac{1 - \lambda(k)}{1 + \lambda(k)} \right| \leqq \exp\left(-2 \sum_{k=1}^{n} \frac{1}{\lambda(k)}\right).$$

In the "unseparated case", $\lambda(k) - \lambda(k-1) < 2$ it is shown that $(\sum_{k=1}^{n} \lambda_k)^{-1/2}$ is the approximation index (which, of course, is determined only to within a multiplicative constant).

The twelfth and final chapter is again concerned with Jackson's theorem, this time in the setting of a translation invariant Banach space of 2π-periodic functions on the real line. There follow some brief bibliographical notes, a bibliography, an index of symbols and an index.

The authors have achieved their intention of introducing the reader to polynomial approximation, leading him through some essential classical material and bringing him quickly to the active frontier of the subject. The presentation is strongly computational and, even in familiar areas, full of innovations and elegant shortcuts which will appeal to professionals in the field as well as to students. The tone of the exposition is that of extremely relaxed informal conversation trailing off occasionally into limp incoherence. There are many minor typographical errors, none of serious concern to a careful reader.

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