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ORBIT STRUCTURE OF THE EXCEPTIONAL HERMITIAN SYMMETRIC SPACES. II

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This note describes results on the orbit structure of the exceptional hermitian symmetric space $E_6/SO(10) \cdot SO(2)$ analogous to those obtained for the space $E_7/E_6 \cdot SO(2)$ in [2].

1. **J. Tits' construction of the complex Lie algebra $\mathfrak{E}_6$.** Let $\mathcal{A}$ be the algebra $\mathbf{C} \oplus \mathbf{C}$ with componentwise multiplication, and define the trace of an element of $\mathcal{A}$ to be the sum of its two components. As in [2], let $\mathcal{J}$ be the 27-dimensional Jordan algebra of hermitian 3×3 matrices over the complex Cayley numbers. Write $\mathcal{A}_0$ and $\mathcal{J}_0$ for the subsets of $\mathcal{A}$ and $\mathcal{J}$ consisting of elements with zero trace. Also let $\mathrm{Der}(\mathcal{J})$ be the Lie algebra of derivations of $\mathcal{J}$ and let $\{L(A)\}(B) = A \circ B$ denote multiplication in $\mathcal{J}$. Now define an anticommutative multiplication $[\ ,\ ]$ on the complex vector space $\mathfrak{g} = (\mathcal{A}_0 \otimes \mathcal{J}_0) + \mathrm{Der}(\mathcal{J})$ by means of the following rules:

(a) $[D, D']$ is the usual commutator for $D, D' \in \mathrm{Der}(\mathcal{J})$.

(b) $[D, a \otimes A] = a \otimes D(A)$ for $a \in \mathcal{A}_0$, $A \in \mathcal{J}_0$, and $D \in \mathrm{Der}(\mathcal{J})$.

(c) $[a \otimes A, b \otimes B] = \tfrac{1}{2}\mathrm{Tr}(ab)[L(A), L(B)]$ for $a, b \in \mathcal{A}_0$ and $A, B \in \mathcal{J}_0$.

Then $\mathfrak{g}$ is the complex Lie algebra $\mathfrak{E}_6$.

If we put $e = (1, -1) \in \mathcal{A}_0$, then $\mathcal{A}_0 = \mathbf{C} \cdot e$, so $\mathfrak{g} = (\mathbf{C} \cdot e \otimes \mathcal{J}_0) + \mathrm{Der}(\mathcal{J})$, and the multiplication in $\mathfrak{g}$ is determined by the single rule

$$[e \otimes A + D, e \otimes A' + D'] = e \otimes \{D(A') - D'(A)\} + [L(A), L(A')] + [D, D']$$

for $A, A' \in \mathcal{J}_0$ and $D, D' \in \mathrm{Der}(\mathcal{J})$.

Let $\mathcal{A}'$ be the set of elements in $\mathcal{A}$ of the form (w, w^*), where w^*

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is the complex conjugate of w. Let J' be the set of matrices in J whose entries are real Cayley numbers. Choose an element (i, j, k) from the set $\{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$ and (in the notation of [2, §5]) let J'' be the subset of J whose entries are of the form $\xi_i E_i + \xi_j E_j + \xi_k E_k + F_i\{x_i\} + F_j\{x_j\} + F_k\{x_k\}$, where ξ_i, ξ_j, ξ_k are real numbers and $\sqrt{-1}x_i, \sqrt{-1}x_j, x_k$ are real Cayley numbers. If we substitute A' and J' for A and J in the above construction, we obtain a compact real form $\mathfrak{g}_c$ of $\mathfrak{g}$. If instead we substitute A' and J'', we obtain a noncompact real form $\mathfrak{g}_0$ of $\mathfrak{g}$ with Cartan index -14.

All of the above results come from [7].

2. **Conventions.** Put $X_c = E_6/SO(10)\cdot SO(2)$ and let X_0 be the noncompact dual of X_c. Define $\mathfrak{k}$ and $\mathfrak{m}_0$ as in [2, §1], write $X_0 = G_0/K$ and $X_c = G_c/K$ as in [2, §2], and define $\mathfrak{m}^+$, $\mathfrak{m}^-$, ξ, Ω, and f_u (for $u \in \mathfrak{m}^+$) as in [2, §§3, 4].

Let K denote the subspace of J consisting of elements of the form $F_i\{x_i\} + F_j\{x_j\}$, where x_i and x_j are complex Cayley numbers.

3. **Realization of X_0 as a bounded symmetric domain.** Using the construction of $\mathfrak{g}_0$ and $\mathfrak{g}_c$ in §1, we find that $\mathfrak{m}^+$ and $\mathfrak{m}^-$ are isomorphic to K. If we identify $\mathfrak{m}^+$ and $\mathfrak{m}^-$ with K, then for each $u \in \mathfrak{m}^+ = K$, f_u can be viewed as the endomorphism of $\mathfrak{m}^- = K$ defined by

$$f_u = 2\{L(u \circ u^*) - [L(u), L(u^*)]\}.$$

Hence by Langlands' theorem (cf. [2, §4]), we obtain

THEOREM 1. $\Omega = \{u \in K: L(u \circ u^*) - [L(u), L(u^*)] < I\}$.

As in [2], this description can be compared with those obtained by M. Koecher [6] and M. Ise [4], [5], who used different methods. Renumbering so that the choices of (i, j, k) coincide, we obtain

THEOREM 2. *The three descriptions of Ω are identical as point sets.*

4. **Notations.** If c and d are nonnegative integers such that $0 \leqslant c + d \leqslant 16$, let $K(c, d)$ denote the set of matrices u in K such that f_u has c eigenvalues < 2 and d eigenvalues > 2 (hence $16 - c - d$ eigenvalues $= 2$). Then $\Omega = K(16, 0)$ by Theorem 1.

Let $\{e_0, e_1, \cdots, e_7\}$ be the usual basis for the complex Cayley numbers. Define another basis $\{e'_0, e'_1, \cdots, e'_7\}$ by

$$e'_{2t} = \tfrac{1}{2}(\sqrt{-1}\,e_t + e_{7-t}) \quad \text{and} \quad e'_{2t+1} = \tfrac{1}{2}(-\sqrt{-1}\,e_t + e_{7-t}), \qquad 0 \leqslant t \leqslant 3.$$

Let Δ be the set of matrices of K of the form $F_i\{re'_0 + se'_1\}$, where r and s are real numbers. If a and b are nonnegative integers with $0 \leqslant a + b \leqslant 2$, let $\Delta(a, b)$ denote the $\mathrm{Ad}(K)$-orbit of the set of matrices $F_i\{re'_0 + se'_1\}$ in Δ such that a of the numbers r^2, s^2 are < 1 and b of them are > 1.

5. **The G_0-orbit structure of $X_c = E_6/SO(10) \cdot SO(2)$.** The following result is proved by means of general theory from [9] and a study of the eigenvalues of f_u for $u \in K$.

THEOREM 3. *The pullbacks under ξ of the G_0-orbits on X_c are the sets $\Delta(a, b)$, where a and b are nonnegative integers such that $0 \leqslant a + b \leqslant 2$. These sets can be described in terms of the eigenvalues of f_u, $u \in K$, as follows*:

$$\Delta(2, 0) = K(16, 0),$$

$$\Delta(0, 2) = K(0, 16) \cup K(4, 12) \cup K(8, 8) \cup K(0, 12) \cup K(4, 8) \cup K(0, 8),$$

$$\Delta(1, 1) = K(15, 1) \cup K(9, 7) \cup K(5, 11) \cup K(5, 7) \cup K(9, 1) \cup K(5, 1),$$

$$\Delta(1, 0) = K(15, 0),$$

$$\Delta(0, 1) = K(8, 7) \cup K(4, 11) \cup K(4, 7), \quad \textit{and}$$

$$\Delta(0, 0) = K(8, 0).$$

Let $S(a, b)$ denote the G_0-orbit on X_c whose pullback under ξ is $\Delta(a, b)$. Then

(a) *The open G_0-orbits on X_c are $X_0 = S(2, 0), S(1, 1)$, and $S(0, 2)$.*

(b) *The topological boundaries in X_c of the open orbits $S(2, 0)$, $S(1, 1)$, and $S(0, 2)$ are $S(1, 0) \cup S(0, 0)$, $S(1, 0) \cup S(0, 1) \cup S(0, 0)$, and $S(0, 1) \cup S(0, 0)$ respectively.*

(c) *$S(0, 0)$ is the Bergman-Šilov boundary of X_0 in X_c, the unique closed G_0-orbit on X_c.*

(d) *$S(a', b')$ is in the closure of $S(a, b)$ if and only if $a' \leqslant a$ and $b' \leqslant b$.*

(Part (d) of Theorem 3 in [2] is incorrect. It should be identical to part (d) of the above theorem.)

6. **Holomorphic arc components.** Theorem 4 below is proved by means of results from **[9]** and the eigenvalue analysis mentioned in §5. The determination of the rank 1 holomorphic arc components necessitates some computations involving the orbit structure of the rank 1 hermitian symmetric space $SU(6)/S(U_1 \times U_5)$ (notation as in [**3**, p. 354]), whose noncompact dual has a bounded realization as the open unit ball in $\mathbf{C}^5$.

THEOREM 4. *Let a and b be nonnegative integers with $0 \leqslant a + b \leqslant 2$. Then the holomorphic arc components of the G_0-orbit $S(a, b)$ are symmetric spaces of rank $a + b$ whose pullbacks under ξ are the sets* $\mathrm{Ad}(K) \cdot C(a, b)$, $k \in K$, *where the subset $C(a, b)$ of $\mathfrak{m}^+ = K$ is described for each choice of a and b as follows: Choose (i, j, k) as in §1. Then*

$$C(0, 0) = \{-F_i\{e_7\}\},$$

$$C(1, 0) = \left\{F_i\{-e'_0 + z_1e'_1\} + F_j\{z_2e'_0 + z_3e'_2 + z_4e'_4 + z_5e'_6\}\colon \sum_{m=1}^{5} |z_m|^2 < 1\right\},$$

$$C(0, 1) = \left\{F_i\{-e'_0 + z_1e'_1\} + F_j\{z_2e'_0 + z_3e'_2 + z_4e'_4 + z_5e'_6\}\colon \sum_{m=1}^{5} |z_m|^2 > 1\right\},$$

$$C(a, b) = \Delta(a, b) \quad \text{when } a + b = 2.$$

In particular, the boundary components of X_0 have pullbacks $\mathrm{Ad}(k) \cdot C(a, 0)$ *where $k \in K$ and $a = 0$ or 1.*

Details and complete proofs for the results in this note and in the previous one [**2**] will appear elsewhere.

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