BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 6, November 1974

THE SPACE OF CLASS α BAIRE FUNCTIONS

BY J. E. JAYNE

Communicated by Jacob Feldman, December 6, 1973

ABSTRACT. Let X, Y be compact Hausdorff spaces and $B_\alpha^*(X)$, $B_\beta^*(Y)$, $0 \leqslant \alpha$, $\beta \leqslant \Omega$ (the first uncountable ordinal), the associated Banach spaces of bounded real-valued Baire functions of classes α and β. If $B_\alpha^*(X) \neq B_\beta^*(X)$ (which is the case if $\alpha \neq \beta$ and X is not dispersed), then $B_\alpha^*(X)$ is neither linearly isometric to $B_\beta^*(Y)$ nor equivalent to $B_\beta^*(Y)$ in several other ways. $B_\Omega^*(X)$ is linearly isometric to $B_\Omega^*(Y)$ if and only if X is Baire isomorphic to Y. For $1 \leqslant \alpha < \Omega$ the maximal ideal space of $B_\alpha^*(X)$ for a nondispersed compact space X is not an F-space.

1. Let X be a compact (more generally, completely regular) Hausdorff space and $C(X)$ the space of continuous real-valued functions on X. Let $B_0(X) = C(X)$, and inductively define $B_\alpha(X)$ for each ordinal $\alpha \leqslant \Omega$ (Ω denotes the first uncountable ordinal) to be the space of pointwise limits of sequences of functions in $\bigcup_{\xi<\alpha} B_\xi(X)$. Let $B_\alpha^*(X)$ be the space of bounded functions contained in $B_\alpha(X)$. With the pointwise operations $B_\alpha(X)$ and $B_\alpha^*(X)$ are lattice-ordered algebras. With the supremum norm $B_\alpha^*(X)$ is a Banach algebra (see [4, §41]).

The Baire sets of X of multiplicative class α, denoted by $Z_\alpha(X)$, are defined to be the zero sets of functions in $B_\alpha^*(X)$. Those of additive class α, denoted by $CZ_\alpha(X)$, are defined as the complements of sets in $Z_\alpha(X)$. Finally, those of ambiguous class α, denoted by $A_\alpha(X)$, are the sets which are simultaneously in $Z_\alpha(X)$ and $CZ_\alpha(X)$. With the set-theoretic operations of union and intersection, $A_\alpha(X)$ is a Boolean algebra for each $\alpha \leqslant \Omega$. The sets of exactly ambiguous class α, denoted by $EA_\alpha(X)$, are those in $A_\alpha(X)\backslash\bigcup_{\xi<\alpha} A_\xi(X)$. The sets of exactly additive and exactly multiplicative class α are defined analogously. The class of all Baire subsets of X is $Z_\Omega(X)$.

AMS (MOS) subject classifications (1970). Primary 06A65, 26A21, 28A05, 46E15, 46E30, 46J10, 54C50, 54H05; Secondary 04A15.

Key words and phrases. Banach space, Baire function, Baire class.

A topological space is called realcompact if it is homeomorphic to a closed subset of a product of real lines.

THEOREM 1. *If X and Y are compact (more generally, realcompact) spaces, then every Boolean algebra isomorphism f of $A_\alpha(X)$ onto $A_\beta(Y)$, $1 \leqslant \alpha, \beta \leqslant \Omega$, is induced by a point map ϕ of X onto Y; that is, there exists a one-to-one map ϕ of X onto Y such that $\phi[B] = f(B)$ for each $B \in A_\alpha(X)$.*

PROOF OUTLINE. Consider the compact set, denoted by X_α, of non-zero multiplicative linear functionals on $B_\alpha^*(X)$ with the weak star topology. It follows from the fact that for each pair of disjoint sets $B_1, B_2 \in Z_\alpha(X)$, there is an $A \in A_\alpha(X)$ with $B_1 \subseteq A \subseteq X \backslash B_2$, that X_α has a base of clopen (closed and open) sets. Since the Boolean algebra of clopen sets of X is isomorphic to $A_\alpha(X)$, the Stone space of $A_\alpha(X)$ is homeomorphic to X_α.

The canonical embedding of X into X_α which assigns a point in X to the evaluation functional at that point maps X onto a dense subset of X_α. The induced topology on X from X_α is discrete if and only if every point in X is a G_δ. The space X_α may thus be considered as a compactification of X with the topology having $Z_0(X)$ as a base. From this point of view, each $f \in B_\alpha^*(X)$ has a unique extension to a $\hat{f} \in C(X_\alpha)$, and the map Φ: $B_\alpha^*(X) \rightarrow C(X_\alpha)$ defined by $\Phi(f) = \hat{f}$ is an algebra isomorphism onto $C(X_\alpha)$. Details concerning the space X_α are contained in [5].

A filter $\mathcal{F}$ of sets in $A_\alpha(X)$ $(Z_\alpha(X))$ is said to have the CIP (countable intersection property) if for each countable family $\{C_n\} \subseteq \mathcal{F}$ there is a $C \in A_\alpha(X)$ (respectively $Z_\alpha(X)$) such that $C \subseteq \bigcap_{n=1}^\infty C_n$. A filter $\mathcal{F}$ in $A_\alpha(X)$ $(Z_\alpha(X))$ is said to be fixed if $\bigcap\{F\colon F \in \mathcal{F}\} \neq \varnothing$.

For any completely regular spaces X and Y, if f is a Boolean algebra isomorphism of $A_\alpha(X)$ onto $A_\beta(Y)$ and $\mathcal{M}$ is a maximal filter in $A_\alpha(X)$ with the CIP, then $f[\mathcal{M}]$ is a maximal filter in $A_\beta(Y)$ with the CIP.

If X is realcompact, then every maximal filter with the CIP in $A_\alpha(X)$, $\alpha \geqslant 1$, is fixed. To see this let $\mathcal{M} \subseteq A_\alpha(X)$ be a maximal filter with the CIP. Then, since each element of $Z_\alpha(X)$ is the countable intersection of elements in $A_\alpha(X)$, $\mathcal{M}_\delta$ (the family of countable intersections of sets in $\mathcal{M}$) is a maximal filter with the CIP in $Z_\alpha(X)$. Thus, since X is realcompact and each set in $Z_\alpha(X)$ is obtainable from $Z_0(X)$ by Souslin's operation (A), $\mathcal{M}_\delta$ is fixed. Thus $\mathcal{M}$ is fixed. Here we have used the following set-theoretic result due to Z. Frolik [2]: Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be families of subsets of a set

X which are closed under countable intersections, and let $\mathcal{M}(\mathcal{H}_i)$, $i = 1, 2$, denote the set of free maximal filters in $\mathcal{H}_i$ with the CIP. If $\mathcal{H}_1 \subseteq \mathcal{H}_2$ and every $H \in \mathcal{H}_2$ is a Souslin-$\mathcal{H}_1$ set (that is, can be represented in the form

$$H = \bigcup_{i_1, i_2, \cdots} \bigcap_{n=1}^{\infty} H_{i_1, \dots, i_n}, \qquad H_{i_1, \dots, i_n} \in \mathcal{H}_1$$

where the union is over all sequences of positive integers $(i_1, i_2, \cdots)$), then the map $\mathcal{M} \rightarrow \mathcal{M} \cap \mathcal{H}_1$, $\mathcal{M} \in \mathcal{M}(\mathcal{H}_2)$ is one-to-one onto $\mathcal{M}(\mathcal{H}_1)$.

From this it follows that if X and Y are realcompact, $\alpha, \beta \geqslant 1$, and f is a Boolean algebra isomorphism of $A_\alpha(X)$ onto $A_\beta(Y)$, then the induced homeomorphism ϕ of their Stone spaces, namely X_α and Y_β, maps X onto Y; that is, $\phi[X] = Y$. Also for each $B \in A_\alpha(X)$, $\phi[B] = f(B)$. This completes the proof.

2. Let X and Y be completely regular spaces. A Baire isomorphism of class $(\alpha, \beta; \gamma, \delta)$ of X onto Y is a one-to-one map f of X onto Y such that

$$f[Z_\alpha(X)] \subseteq Z_\beta(Y) \quad \text{and} \quad f^{-1}[Z_\delta(Y)] \subseteq Z_\gamma(X).$$

THEOREM 2. *If X and Y are compact (more generally, realcompact) spaces and $\alpha, \beta \geqslant 1$, then the following are equivalent:*

(1) *There exists a Baire isomorphism of class $(\alpha, \beta; \alpha, \beta)$ of X onto Y.*

(2) *$B_\alpha^*(X)$ is linearly isometric to $B_\beta^*(Y)$.*

(3), (4), (5), (6) *$B_\alpha^*(X)$ is isometric (ring, lattice, multiplicative semigroup isomorphic) to $B_\beta^*(Y)$.*

(6), (7), (8), (9) *$B_\alpha(X)$ is ring (lattice, multiplicative semigroup) isomorphic to $B_\beta(Y)$.*

PROOF OUTLINE. (2) $\Rightarrow$ (1). Since $B_\alpha^*(X)$ and $B_\beta^*(Y)$ are linearly isometric to $C(X_\alpha)$ and $C(Y_\beta)$ respectively, X_α and Y_β are homeomorphic. By Theorem 1 such a homeomorphism induces a Baire isomorphism of class $(\alpha, \beta; \alpha, \beta)$ of X onto Y.

All of the other nontrivial implications follow similarly.

REMARK. Since for a completely regular space X every $f \in B_\alpha^*(X)$ $(B_\alpha(X))$ has a unique extension to a $\hat{f} \in B_\alpha^*(\nu X)(B_\alpha(X))$, where νX denotes the Hewitt realcompactification of X [6], it follows from Theorem 2 that for completely regular spaces X and Y, parts (2) through (9) of Theorem 2 are equivalent, and these are equivalent to the existence of a Baire isomorphism of class $(\alpha, \beta; \alpha, \beta)$ of νX onto νY. More generally yet, Theorems 1 and 2 may be phrased in terms of zero-set spaces and suitably defined 0-dimensional zero-set spaces, their realcompactifications, and their associated function spaces (see [3]).

3. Recently F. Dashiell [1] has shown that if X is an uncountable compact metric space, then for $\alpha \neq \beta$, $B_\alpha^*(X)$ is not linearly isometric to $B_\beta^*(X)$, which may be thought of as strengthening the classical result that for $\alpha < \beta$, $B_\alpha^*(X)$ is a proper subspace of $B_\beta^*(X)$.

A compact space is called dispersed if it contains no nonempty perfect subsets. It is known that a compact space X contains a nonempty perfect subset if and only if for each $\alpha < \Omega$, $B_\alpha^*(X)$ is a proper subspace of $B_{\alpha+1}^*(X)$, and if and only if $B_2^*(X)\backslash B_1^*(X) \neq \varnothing$ (see [5] and [6]). Part of this also follows from the next theorem, since a nondispersed compact space admits a continuous map onto the unit interval.

THEOREM 3. *If X is a compact space, $\alpha \geqslant 0$, f a continuous real-valued map on X, and $B \in EA_\alpha(f[X])$, then $f^{-1}[B] \in EA_\alpha(X)$. The same holds for exactly additive and exactly multiplicative classes.*

THEOREM 4. *If f is a continuous map of a compact space X onto a compact space Y, then for $\alpha = 0, 1, 2$ or $\alpha \geqslant \omega_0$, $B \in EA_\alpha(Y)$ implies that $f^{-1}[B] \in EA_\alpha(X)$, and for $2 < \alpha < \omega_0$, $B \in A_\alpha(Y)$ implies that $f^{-1}[B] \in EA_{\alpha-1}(X) \cup EA_\alpha(X)$. The same holds for exactly additive and exactly multiplicative classes.*

THEOREM 5. (1) *Let X and Y be compact spaces and suppose that either X or Y is not dispersed. For $0 \leqslant \alpha < \beta \leqslant \Omega$, $B_\alpha^*(X)$ is not linearly isometric to $B_\beta^*(Y)$.*

(2) *If X and Y are infinite dispersed compact spaces, then $B_0^*(X)$ is not linearly isometric to $B_1^*(Y)$. (Note that $B_1^*(X) = B_2^*(X)$.)*

PROOF OUTLINE. (1) Suppose $\alpha < \beta$, that X is not dispersed, and that $B_\alpha^*(X)$ is linearly isometric to $B_\beta^*(Y)$. Then by Theorem 2 there is a Baire isomorphism ϕ of Y onto X of class $(\beta, \alpha; \beta, \alpha)$. Thus there is a ring isomorphism Φ of $B_\Omega(X)$ onto $B_\Omega(Y)$ such that $\Phi[B_\alpha(X)] = B_\beta(Y)$ defined by $\Phi(h)(y) = h(\phi(y))$ for all $h \in B_\alpha(X)$ and $y \in Y$. Let $f: X \to [0, 1]$ be a continuous map onto the unit interval. Let $\{g_n: n = 1, 2, \cdots\} \subseteq C(Y)$ be such that $\Phi(f)$ is contained in the smallest class of functions containing $\{g_n: n = 1, 2, \cdots\}$ and closed under pointwise sequential limits.

Consider the map $\Psi: Y \to \mathbf{R}^N$ defined by $\Psi(y) = (g_1(y), g_2(y), \cdots)$. Then the Baire isomorphism ϕ of Y onto X induces a Baire measurable map $\widetilde{\phi}$ of $\Psi[Y]$ onto $[0, 1]$. There is a Cantor set $C \subseteq \Psi[Y]$ such that $\widetilde{\phi}$ restricted to C is a homeomorphism [7, p. 444], since $\widetilde{\phi}$ is continuous apart from a set of first category [7, p. 400]. Thus there is a set $B \in EA_\beta(\widetilde{\phi}[C]) \subseteq EA_\beta([0, 1])$. By Theorem 3, $\Psi^{-1}[\widetilde{\phi}^{-1}[B]] \in EA_\beta(Y)$ and $f^{-1}[B] \in EA_\beta(X)$.

This implies that the characteristic function

$$\chi_{\Psi^{-1}[\widetilde{\phi}^{-1}[B]]} \in B^*_\beta(Y) \setminus \bigcup_{\xi<\beta} B^*_\xi(Y).$$

But since $\Phi^{-1}(\chi_{\Psi^{-1}[\widetilde{\phi}^{-1}[B]]}) = \chi_{f^{-1}[B]}$, it follows that $\chi_{f^{-1}[B]} \in B^*_\alpha(X)$. This contradiction completes the proof.

(2) This follows from the fact that X_1 and Y_1 contain nonempty compact perfect subsets and X and Y do not.

4. A topological space X is called an F-space if for each disjoint pair $C_1, C_2 \in CZ_0(X)$ there is a disjoint pair $Z_1, Z_2 \in Z_0(X)$ such that $C_1 \subseteq Z_1$ and $C_2 \subseteq Z_2$.

LEMMA. *Let X be any topological space and $\alpha \geqslant 1$. If $Z_1, Z_2 \in Z_\alpha(X)$ are disjoint, then there is a set $A \in A_\alpha(X)$ such that $Z_1 \subseteq A \subseteq X \setminus Z_2$.*

THEOREM 6. *If X is a nondispersed compact space and $1 \leqslant \alpha < \Omega$, then there exist disjoint sets $C_1, C_2 \in CZ_\alpha(X)$ such that there does not exist a set $A \in A_\alpha(X)$ with $C_1 \subseteq A \subseteq X \setminus C_2$. Consequently, X_α, the Stone space of $A_\alpha(X)$, is not an F-space.*

Remarks. (1) For any space X the Stone space of $A_\Omega(X) = (= Z_\Omega(X))$ is an F-space.

(2) For uncountable compact metric spaces a stronger result than Theorem 5 is obtained in [1].

REFERENCES

1. F. K. Dashiell, *Isomorphism problems for the Baire classes,* Pacific J. Math. (to appear).

2. Z. Frolík, *Realcompactness is a Baire-measurable property,* Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 19 (1971), 617–621. MR 46 #312.

3. H. Gordon, *Rings of functions determined by zero sets*, Pacific J. Math. **36** (1971), 133–157.

4. F. Hausdorff, *Set theory*, Chelsea, New York, 1957. MR **19**, 111.

5. J. E. Jayne, *Spaces of Baire functions, Baire classes, and Souslin sets*, Ph. D. Dissertation, Columbia Univ., New York, 1971.

6. ———, *Spaces of Baire functions*. I, Ann. Inst. Fourier (Grenoble) (to appear).

7. K. Kuratowski, *Topology*. Vol. I, Academic Press, New York, 1966.

DEPARTMENT OF MATHEMATICS, UNIVERSITY COLLEGE, LONDON, ENGLAND (Current address)

INSTITUTE OF MATHEMATICS, CZECHOSLOVAK ACADEMY OF SCIENCES, PRAGUE, CZECHOSLOVAKIA