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COMBINATORIAL AND CONTINUOUS HODGE THEORIES

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0. Let K be a finite simplicial complex. Eckmann (see [**1**]) observed that any inner product in cochain spaces $C^q(K; \boldsymbol{R})$ gives rise to a combinatorial Hodge theory. The purpose of this note is to announce that if K is a smooth triangulation of a compact, oriented Riemannian manifold X, then the combinatorial Hodge theory (for a suitable choice of inner product in cochain spaces) is an approximation of the Hodge theory of forms on X. We wish to thank L. Bers, H. Garland, and I. M. Singer for their help in our research.

1. **Whitney map and definition of inner product.** Let Λ^q and $L^2\,\Lambda^q$ denote the spaces C^∞ and L^2 q-forms on X respectively. Whitney (see [**2**]) defined a linear mapping $W: C^q(K; \boldsymbol{R}) \to L^2\,\Lambda^q$, as follows. Let $\sigma = [p_0, \cdots, p_q]$ be a q-simplex of K and let $\mu_0, \cdots, \mu_q$ be the barycentric coordinates corresponding to $p_0, p_1, \cdots, p_q$ respectively; then

$$W\sigma = q! \sum_{i=0}^{q} (-1)^i \mu_i \, d\mu_0 \wedge \cdots \wedge d\mu_{i-1} \wedge d\mu_{i+1} \wedge \cdots \wedge d\mu_q.$$

This defines W uniquely since q-simplexes span $C^q(K; \boldsymbol{R})$. The μ_i's are C^∞ on every closed simplex of K which allows us to apply the exterior derivative d in the formula above.

Let c, c' be two q-cochains. We set $(c, c') = \int_X Wc \wedge * Wc'$. $(\ ,\ )$ is obviously symmetric and positive semidefinite. It actually turns out to be an inner product.

2. **Approximation theorem.** Let S_nK be the nth standard subdivision of K (see [**2**]). We write $C_n^q = C^q(S_nK; \boldsymbol{R})$. For every nonnegative integer n the Whitney map $W_n: C_n^q \to L^2\,\Lambda^q$ induces an inner product in C_n^q as above. Let $R_n: \Lambda^q \to C_n^q$ be the de Rham map defined by integration of forms over simplicial chains of S_nK.

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Let $\| \ \|_p$ be the norm in $\Lambda^q T^*(X)_p$ induced by the Riemannian metric. Let $\| \ \|$ be the norm in $L^2 \Lambda^q$. Let η_n be the mesh of $S_n K$. Of course, $\lim_{n\to\infty} \eta_n = 0$. We can now state the approximation theorem.

THEOREM 1. *Let f be a C^∞ q-form on X. There exists a constant C_f such that for every nonegative integer n*

$$\|f(p) - W_n R_n f(p)\|_p \leqq C_f \cdot \eta_n$$

almost everywhere on X.

COROLLARY. *There exists a constant c_f such that $\|f - W_n R_n f\| \leqq c_f \cdot \eta_n$ for all nonnegative integers n.*

3. **Combinatorial Hodge theory and passage to the limit.** Let $d_n : C_n^q \to C_n^{q+1}$ be the simplicial coboundary. Let δ_n be the adjoint of d_n with respect to the inner product described above. We set $\Delta_n = d_n\delta_n + \delta_n d_n$ and let H_n^q be the kernel of Δ_n acting on C_n^q. C_n^q has an orthogonal decomposition (Hodge decomposition)

$$C_n^q = d_n C_n^{q-1} \oplus H_n^q \oplus \delta_n C_n^{q+1}.$$

Moreover $H_n^q = \{c \in C_n^q \mid d_n c = \delta_n c = 0\}$ and H_n^q is isomorphic to $H^q(X; \boldsymbol{R})$, the qth cohomology group of X.

THEOREM 2. *Let $f = dg + h + \delta k$ be the Hodge decomposition of a C^∞ q-form f. Let $R_n f = d_n g_n + h_n + \delta_n k_n$ be the Hodge decomposition of the cochain $R_n f$. There exists a constant c_f such that, for $n = 1, 2, \cdots$,*

$$\|W_n d_n g_n - dg\| \leqq c_f \cdot \eta_n,$$

$$\|W_n h_n - h\| \leqq c_f \cdot \eta_n,$$

$$\|W_n \delta_n k_n - \delta k\| \leqq c_f \cdot \eta_n.$$

Let $0 = \lambda_0 < \lambda_1 \leqq \lambda_2 \leqq \cdots \to \infty$ be the sequence of eigenvalues of the Laplacian Δ acting on C^∞ functions on X. For an integer $n \geqq 0$, let $d(n) = \dim C_n^0$ and let $0 = \lambda_0^{(n)} < \lambda_1^{(n)} \leqq \lambda_2^{(n)} \leqq \cdots \leqq \lambda_{d(n)}^{(n)}$ be the sequence of eigenvalues of combinatorial Laplacian Δ_n acting on C_n^0.

THEOREM 3. *For every positive integer i there exists a constant c_i such that, if $i \leqq d(n)$, $\lambda_i^{(n)} - C_i \eta_n \leqq \lambda_i \leqq \lambda_i^{(n)}$. In particular, $\lim_{n\to\infty} \lambda_i^{(n)} = \lambda_i$.*

We conjecture that Theorem 3 is true for *all* dimensions $q = 0, 1, 2, \cdots, \dim X$.

4. **Generalizations.** The above technique and results can be generalized in two ways. On one hand, we can replace X by a manifold with boundary and consider forms satisfying certain boundary conditions and relative

cochains. On the other hand, our results generalize to forms and cochains with values in a vector bundle induced by an orthogonal representation of the fundamental group of X. Results analogous to Theorems 1, 2, 3 hold in both cases.

REFERENCES

1. B. Eckmann, *Harmonische Funktionen und Randvertaufgaben in einem komplex*, Comment. Math. Helv. **17** (1945), 240–255. MR **7**, 138.

2. H. Whitney, *Geometric integration theory*, Princeton Univ. Press, Princeton, N.J., 1957. MR **19**, 309.

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