

COMBINATORIAL AND CONTINUOUS HODGE THEORIES

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0. Let K be a finite simplicial complex. Eckmann (see [1]) observed that any inner product in cochain spaces $C^q(K; \mathbf{R})$ gives rise to a combinatorial Hodge theory. The purpose of this note is to announce that if K is a smooth triangulation of a compact, oriented Riemannian manifold X , then the combinatorial Hodge theory (for a suitable choice of inner product in cochain spaces) is an approximation of the Hodge theory of forms on X . We wish to thank L. Bers, H. Garland, and I. M. Singer for their help in our research.

1. **Whitney map and definition of inner product.** Let Λ^q and $L^2 \Lambda^q$ denote the spaces C^∞ and L^2 q -forms on X respectively. Whitney (see [2]) defined a linear mapping $W: C^q(K; \mathbf{R}) \rightarrow L^2 \Lambda^q$, as follows. Let $\sigma = [p_0, \dots, p_q]$ be a q -simplex of K and let μ_0, \dots, μ_q be the barycentric coordinates corresponding to p_0, p_1, \dots, p_q respectively; then

$$W\sigma = q! \sum_{i=0}^q (-1)^i \mu_i d\mu_0 \wedge \dots \wedge d\mu_{i-1} \wedge d\mu_{i+1} \wedge \dots \wedge d\mu_q.$$

This defines W uniquely since q -simplexes span $C^q(K; \mathbf{R})$. The μ_i 's are C^∞ on every closed simplex of K which allows us to apply the exterior derivative d in the formula above.

Let c, c' be two q -cochains. We set $(c, c') = \int_X Wc \wedge * Wc'$. $(,)$ is obviously symmetric and positive semidefinite. It actually turns out to be an inner product.

2. **Approximation theorem.** Let $S_n K$ be the n th standard subdivision of K (see [2]). We write $C_n^q = C^q(S_n K; \mathbf{R})$. For every nonnegative integer n the Whitney map $W_n: C_n^q \rightarrow L^2 \Lambda^q$ induces an inner product in C_n^q as above. Let $R_n: \Lambda^q \rightarrow C_n^q$ be the de Rham map defined by integration of forms over simplicial chains of $S_n K$.

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Let $\| \cdot \|_p$ be the norm in $\Lambda^q T^*(X)_p$ induced by the Riemannian metric. Let $\| \cdot \|$ be the norm in $L^2 \Lambda^q$. Let η_n be the mesh of $S_n K$. Of course, $\lim_{n \rightarrow \infty} \eta_n = 0$. We can now state the approximation theorem.

THEOREM 1. *Let f be a C^∞ q -form on X . There exists a constant C_f such that for every nonnegative integer n*

$$\|f(p) - W_n R_n f(p)\|_p \leq C_f \cdot \eta_n$$

almost everywhere on X .

COROLLARY. *There exists a constant c_f such that $\|f - W_n R_n f\| \leq c_f \cdot \eta_n$ for all nonnegative integers n .*

3. Combinatorial Hodge theory and passage to the limit. Let $d_n: C_n^q \rightarrow C_n^{q+1}$ be the simplicial coboundary. Let δ_n be the adjoint of d_n with respect to the inner product described above. We set $\Delta_n = d_n \delta_n + \delta_n d_n$ and let H_n^q be the kernel of Δ_n acting on C_n^q . C_n^q has an orthogonal decomposition (Hodge decomposition)

$$C_n^q = d_n C_n^{q-1} \oplus H_n^q \oplus \delta_n C_n^{q+1}.$$

Moreover $H_n^q = \{c \in C_n^q | d_n c = \delta_n c = 0\}$ and H_n^q is isomorphic to $H^q(X; \mathbb{R})$, the q th cohomology group of X .

THEOREM 2. *Let $f = dg + h + \delta k$ be the Hodge decomposition of a C^∞ q -form f . Let $R_n f = d_n g_n + h_n + \delta_n k_n$ be the Hodge decomposition of the cochain $R_n f$. There exists a constant c_f such that, for $n = 1, 2, \dots$,*

$$\|W_n d_n g_n - dg\| \leq c_f \cdot \eta_n,$$

$$\|W_n h_n - h\| \leq c_f \cdot \eta_n,$$

$$\|W_n \delta_n k_n - \delta k\| \leq c_f \cdot \eta_n.$$

Let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \rightarrow \infty$ be the sequence of eigenvalues of the Laplacian Δ acting on C^∞ functions on X . For an integer $n \geq 0$, let $d(n) = \dim C_n^0$ and let $0 = \lambda_0^{(n)} < \lambda_1^{(n)} \leq \lambda_2^{(n)} \leq \dots \leq \lambda_{d(n)}^{(n)}$ be the sequence of eigenvalues of combinatorial Laplacian Δ_n acting on C_n^0 .

THEOREM 3. *For every positive integer i there exists a constant c_i such that, if $i \leq d(n)$, $\lambda_i^{(n)} - C_i \eta_n \leq \lambda_i \leq \lambda_i^{(n)}$. In particular, $\lim_{n \rightarrow \infty} \lambda_i^{(n)} = \lambda_i$.*

We conjecture that Theorem 3 is true for all dimensions $q = 0, 1, 2, \dots, \dim X$.

4. Generalizations. The above technique and results can be generalized in two ways. On one hand, we can replace X by a manifold with boundary and consider forms satisfying certain boundary conditions and relative

cochains. On the other hand, our results generalize to forms and cochains with values in a vector bundle induced by an orthogonal representation of the fundamental group of X . Results analogous to Theorems 1, 2, 3 hold in both cases.

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