BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 5, September 1974

DIFFERENTIABLE Z_p ACTIONS ON HOMOTOPY SPHERES[1]

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Communicated by Glen E. Bredon, January 21, 1974

The results of [**4**] proved that many exotic spheres do not admit smooth actions of relatively high-dimensional compact Lie groups (all group actions considered in this paper are assumed to be effective). It was clear that stronger results should hold in certain cases, and this was confirmed in [**5**]. A notable feature of [**5**] is the use of nonexistence theorems for certain smooth circle actions to prove nonexistence theorems for large Lie group actions. These theorems for circle actions actually reflect much stronger nonexistence results for smooth Z_p actions, the proofs of which are outlined in this paper.

Recently H. B. Lawson and S. T. Yau obtained other (frequently much stronger) nonexistence theorems for connected *nonabelian* actions on exotic spheres using differential-geometric methods and results of N. Hitchin. Since our methods and [**2**] readily yield nonexistence theorems for toral actions, in some sense the topological methods of this paper are complementary to these geometric methods.

As an illustration of the sort of results obtainable by our methods, we give examples of exotic spheres with no effective smooth torus actions.

1. **Normal invariants of homology equivalences.** A *homotopy smoothing* of a smooth manifold M is a pair (X, f) such that $f: X \rightarrow M$ is a homotopy equivalence; a fundamental construction assigns to each homotopy smoothing a normal invariant $\eta(X, f) \in [M, F/O]$. In fact, normal invariants are definable for pairs (X, f) where f is merely a homology equivalence with respect to a subring of the rationals.[3] All such subrings have the form Z_l, the integers with inverses of all primes *not* in l adjoined, and the generalized normal invariant takes its values in $[M, F/O_l] \cong [M, F/O]_l$ (see [**8**, Chapter II] for the relevant localization theory). Many formal properties of ordinary normal invariants which are useful in calculation

AMS (MOS) subject classifications (1970). Primary 57E15, 57E25.

[1] Summary of results.

[2] Partially supported by NSF Grants GP-36418X and GP-19530A1-2.

[3] Various generalizations of normal invariants to homology equivalences have been previously studied by L. Jones, S. Cappell and J. Shaneson, and W. Browder (and probably others).

generalize in an obvious way. We should note that the smooth category is replaceable by the PL and topological categories in this discussion.

2. **Internal normal invariants.** Suppose we are given a smooth orientation-preserving Z_p action on a homotopy sphere Σ^{n+2k}; assume the fixed point set K^n has codimension at least 4, and let S be an invariant $(2k-1)$-sphere in Σ which links K once. Since K is a Z_p-homology sphere by a classical theorem of P. A. Smith, it follows that the induced inclusion $S/Z_p \subseteq \Sigma - K/Z_p$ is a Z_p-homology equivalence and hence a homotopy equivalence of the localizations of these spaces at p. The inverse of this homotopy equivalence induces an equivariant fiber homotopy trivialization of the equivariant fiberwise localization [**8**, Chapter IV] of the equivariant normal bundle of K. In homotopy-theoretic terms, this object corresponds to an element of the set of homotopy classes

$$[K, F_{Z_p}(V)_{(p)}/C_{Z_p}(V)], \tag{2.1}$$

where V is the Z_p-vector space of normal vectors at a fixed point, $C_{Z_p}(V)$ is its orthogonal centralizer, and $F_{Z_p}(V)$ is the space of equivariant self-maps of the unit sphere in V (compare [**1**]). Call this class the *internal normal invariant* of the action.

Every element of (2.1) determines a $Z_{(p)}$-homology equivalence from some S/Z_p-bundle over K^n to $S^n \times (S/Z_p)_{(p)}$. The ideas discussed in §1 yield a map φ from (2.1) to $[S^n L_{(p)} \vee S^n, F/O_{(p)}]$, where $L = S/Z_p$.

LEMMA 2.2. *The space $F_{Z_p}(V)_{(p)}/C_{Z_p}(V)$ is simple, and the map φ factors through $[K, (F_{Z_p}(V)_{(p)}/C_{Z_p}(V))_{(p)}] \cong \pi_n(F_{Z_p}(V)/C_{Z_p}(V))_{(p)}$.*

This result allows us to ignore many of the topological differences between K and the ordinary sphere.

If W is another free Z_p-module, then the diagram

$$\begin{array}{ccc} \pi_n(F_{Z_p}(V)/C_{Z_p}(V))_{(p)} & \xrightarrow{\oplus W} & \pi_n(F_{Z_p}(V \oplus W)/C_{Z_p}(V \oplus W))_{(p)} \\ \downarrow \varphi_0 & & \downarrow \varphi_0 \\ [S^n L \vee S^n, F/O]_{(p)} & \xleftarrow{i^*} & [S^n L' \vee S^n, F/O]_{(p)} \end{array} \tag{2.3}$$

(where L' = orbit space of the unit sphere in $V \oplus W$ and i^* is induced by the inclusion $L \subseteq L'$) commutes. Then we have the following result:

LEMMA 2.4. *Let F_{Z_p} and C_{Z_p} denote the limits of $F_{Z_p}(V \oplus kW)$, $C_{Z_p}(V \oplus kW)$ over all k with respect to Whitney sum (compare* [**1**]). *Then φ_0 depends only upon the image of an element in the stabilized group $\pi_n(F_{Z_p}/C_{Z_p})_{(p)}$.*

The internal normal invariant of a Z_p action gives considerable information about the differential structure on Σ. Let W in (2.3) be two dimensional, and let σ be the map in (2.3) defined by taking Whitney sums with W. Recall that L'/L has the homotopy type of the wedge $S^{2k} \vee S^{2k+1}$.

THEOREM 2.5. *Let ω be the internal normal invariant of the Z_p action on Σ. Then $\varphi(\omega)$ is trivial, and hence $\varphi(\sigma(\omega))$ lies in the image of*

$$[S^n(L'/L), F/O]_{(p)} \cong \pi_{n+2k+1}(F/O)_{(p)} \oplus \pi_{n+2k}(F/O)_{(p)}.$$

This preimage can be chosen so that its π_{n+2k} component is the Pontrjagin-Thom construction on $-\Sigma$.

3. **Applications to special cases.** We first mention a result which follows from techniques developed by G. Bredon.[4]

PROPOSITION 3.0. *Let Σ^m be a homotopy m-sphere admitting a smooth Z_p action with an $(m-2)$-dimensional fixed point set. Then $\Sigma \in p\Gamma_m + bP_{m+1}$.*

Because of this, we can concentrate on the case where the fixed point set has codimension $\geqq 4$.

There is a familiar spectral sequence (see [**3**, Chapter 14]) with

$$E^2_{p,q} = H^{-p}(L'; \pi_q(F/O)) \tag{3.1}$$

which converges to $[S^{p+q}L', F/O]$, where L' is the orbit space in (2.3) and W is arbitrary. Using this spectral sequence and the results of §2 we obtain a necessary condition for the existence of a smooth Z_p action on Σ^{n+2k} with a fixed point set of codimension $2k$.

PROPOSITION 3.2. *If Σ^{n+2k} admits a smooth Z_p action with an n-dimensional fixed point set, then its Pontrjagin-Thom construction in $\pi_{n+2k}(F/O)_{(p)}$ is a permanent cycle in the above spectral sequence* (3.1).

The stable homotopy properties of $K(Z_p, 1)$ together with Proposition 3.2 yield nonexistence theorems such as the following:

THEOREM 3.3. *Let $\alpha_1 \in \pi_{2p-3}$ (the stable stem) generate the p-primary component, let ξ denote a Pontrjagin-Thom construction for Σ and assume $\xi\alpha_1 \notin$ Image J in the stable homotopy groups of spheres. Suppose Z_p acts smoothly on Σ with a $2k$-codimensional fixed point set. Then k is divisible by p.*

Note. Theorem 3.3 implies all the nonexistence theorems for group actions on homotopy spheres in [**4**] and [**5**].

We now specialize to $\xi = \beta_1$, the Pontrjagin-Thom construction on Σ_p.

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[4] See Bredon, *Classification of regular actions of classical groups with three orbit types* (preprint), §5, for the basic construction. I am grateful to Professor Bredon for outlining a proof of 3.0.

Since $\beta_1\alpha_1 \notin \text{Image } J$ by results of Toda, the only codimensions for a Z_p action not ruled out by Theorem 3.3 are $2p, 4p, \cdots, 2(p-2)p$. These restrictions are the best possible, for one can use Theorem 2.5 and methods of [**6**], [**7**], and [**9**] to construct semifree circle actions on Σ_p with each of these numbers as the codimension of the fixed point set (this corrects an assertion at the end of [**6**] concerning such actions). In fact, if $p \geqq 5$, roughly half these actions extend to semifree S^3 actions.

Refinement of the above approach yields similar results for Z_{p^r} actions (p prime); the analog of Proposition 3.2 involves the fixed point set of the Z_p subgroup. This, some results of stable homotopy theory, and equivariant tangent bundle considerations yield further results such as the following:

PROPOSITION 3.4. *Given a smooth circle action on the exotic* 8*-sphere, the fixed point sets of* Z_2 *and* Z_4 *have dimensions* 4 *and* 0 *or* 4 *respectively; conversely, the exotic* 8*-sphere admits smooth circle actions of both types.*

An argument involving Z_4 weight systems gives the following application:

COROLLARY 3.5. *The exotic* 8*-sphere admits no smooth torus actions.*

Details of these and further results of a similar nature will be given in subsequent papers.

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