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THE COMMUTANT OF ANALYTIC TOEPLITZ OPERATORS

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Introduction. The study of analytic Toeplitz operators has been extensive and many of their properties are well known [**1**], [**3**], [**5**]. In [**4**] Nordgren answered questions about the existence of reducing subspaces for certain analytic Toeplitz operators. Since there is a one-to-one correspondence between reducing subspaces for an operator and projections in its commutant, the study of the commutant of analytic Toeplitz operators has an essential relevance to the study of their reducing subspaces. Recently, Deddens and Wong [**2**] provided partial answers to some questions concerning the commutants of analytic Toeplitz operators. The purpose of this paper is to announce results motivated by questions that they raised.

DEFINITIONS AND NOTATION. Let D denote the open unit disk and Γ its boundary, the unit circle. For $\phi \in H^\infty$, T_ϕ is the analytic Toeplitz operator on H^2 defined by $(T_\phi f)(z)=\phi(z)f(z)$. Let $\{T_\phi\}'$ denote the commutant of T_ϕ.

Main results. Analytic Toeplitz operators which are pure isometries, i.e. those induced by inner functions, seem to be of particular importance; and our results center around them. The key to our results is the following function theoretic theorem and its proof.

THEOREM. *Suppose ϕ is a finite Blaschke product with n Blaschke factors and $\psi \in H^\infty$. Then there exists m dividing n and C an open subset of D with $D-C$ discrete such that for each $\lambda \in C$, $\phi^{-1}(\phi(\lambda))\cap\psi^{-1}(\psi(\lambda))$ has precisely m points. Moreover, there exists a finite Blaschke product, I, with m factors such that ϕ and ψ are functions of I.*

THEOREM. *Suppose ϕ is a finite Blaschke product and $\psi \in H^\infty$. Then $\{T_\phi\}'\cap\{T_\psi\}'=\{T_I\}'$, where I is a finite Blaschke product and ϕ and ψ are functions of I.*

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An immediate consequence of the second theorem is an affirmative partial answer to the last question raised in [**2**].

COROLLARY. *Suppose* $\{J_\alpha\}_{\alpha\in A}$ *is a family of* H^∞ *functions such that at least one* J_α *is a finite Blaschke product. Then* $\bigcap_{\alpha\in A}\{T_{J_\alpha}\}'=\{T_I\}'$, *where* I *is inner and each* J_α *is a function of* I.

COROLLARY. *Suppose* $\phi\in H^\infty$ *has inner-outer factorization* $\phi=BF$. *If* B *is a finite Blaschke product, then the following are equivalent*:

(i) $\{T_\phi\}'=\{T_B\}'\cap\{T_F\}'$;

(ii) $\{T_\phi\}'=\{T_I\}'$, *where* I *is a finite Blaschke product and* ϕ *is a function of* I.

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