BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 5, September 1974

ENUMERATION OF PAIRS OF PERMUTATIONS AND SEQUENCES[1]

BY L. CARLITZ, RICHARD SCOVILLE AND THERESA VAUGHAN

Communicated by Paul T. Bateman, January 10, 1974

Let $\pi=(a_1, \cdots, a_n)$ denote a permutation of $Z_n=\{1, 2, \cdots, n\}$. A *rise* of π is a pair a_i, a_{i+1} with $a_i<a_{i+1}$; a *fall* is a pair a_i, a_{i+1} with $a_i>a_{i+1}$. Thus if $\rho=(b_1, \cdots, b_n)$ denotes another permutation of Z_n, the two pairs a_i, a_{i+1}; b_i, b_{i+1} are either both rises, both falls, a rise and a fall or a fall and a rise. We denote these four possibilities by RR, FF, RF, FR, respectively.

Let $\omega(n)$ denote the number of pairs of permutations π, ρ with RR forbidden. More generally let $\omega(n, k)$ denote the number of pairs π, ρ with exactly k occurrences of RR.

THEOREM 1. *We have*

$$\sum_{n=0}^{\infty} \omega(n) \frac{z^n}{n!\,n!} = \left\{\sum_{n=0}^{\infty} (-1)^n \frac{z^n}{n!\,n!}\right\}^{-1}, \tag{1}$$

where $\omega(0)=\omega(1)=1$.

THEOREM 2.

$$\sum_{n=0}^{\infty} \frac{z^n}{n!\,n!} \sum_{k=0}^{n-1} \omega(n, k)x^k = \frac{1-x}{f(z(1-x))-x}, \tag{2}$$

where $f(z)=\sum_{n=0}^{\infty} (-1)^n(z^n/n!n!)$.

The pair π, ρ is said to be *amicable* if RF and FR are both forbidden. Let $\alpha(n)$ denote the number of amicable pairs of Z_n; more generally let $\alpha(n, k)$ denote the number of pairs π, ρ with k total occurrences of RF and FR.

THEOREM 3. *We have*

$$A(z)A(-z) = 1, \tag{3}$$

where $A(z)=\sum_{n=0}^{\infty} \alpha(n)z^n/n!n!$.

AMS (MOS) subject classifications (1970). Primary 05A15, 10A45.
Key words and phrases. Permutations, sequences, generating functions.
[1] Supported in part by NSF grant GP-37924.

Equation (3) is equivalent to $\alpha(0)=1$ and

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2\alpha(k)\alpha(n-k)=0 \qquad (n>0). \tag{4}$$

Unfortunately (4) does not suffice to determine $\alpha(n)$.

THEOREM 4. *We have*

$$1+\sum_{n=1}^{\infty}\frac{x^n}{n!\,n!}\sum_{k=0}^{n-1}\alpha(n,k)y^k=\frac{(1-y)A(x(1-y))}{1-yA(x(1-y))}. \tag{5}$$

We next consider pairs of sequences. The sequence $\sigma=(a_1, a_2, \cdots, a_N)$ $(a_i\in Z_n)$ is said to be of specification $[e]=[e_1, \cdots, e_n]$ if each element in Z_n occurs exactly e_j times, where $e_1+\cdots+e_n=N$. Enumeration of sequences subject to various requirements has been discussed in a number of papers [**1**], [**2**], [**3**], [**4**], [**5**]. The pair a_i, a_{i+1} is a *rise*, *fall* or *level* according as $a_i<a_{i+1}$, $a_i>a_{i+1}$, $a_i=a_{i+1}$ $(i=1, 2, \cdots, N-1)$.

Let $\tau=(b_1, \cdots, b_N)$ denote a sequence of specification $[f]=[f_1, \cdots, f_n]$. Then for the pair σ, τ there are now nine possibilities, namely

$$RR,\ FR,\ LR,\ RF,\ FF,\ LF,\ RL,\ FL,\ LL. \tag{6}$$

Let $Q^{(n)}(r; e,f)$ denote the number of pairs of sequences σ, τ of specification $[e]$, $[f]$, respectively, and with exactly $N-r-1$ occurrences of RR and put

$$Q^{(n)}(x, y, z)=\sum_{e,f,r}Q^{(n)}(r; e,f)x^{e}y^{e}z^{r},$$

where $x^{e}=x_1^{e_1}\cdots x_n^{e_n}$.

THEOREM 5. *We have*

$$Q^{(n)}(x, y, z)=1/D_n, \tag{7}$$

where

$$D_n=1-S_1(x)S_1(y)+(1-z)S_2(x)S_2(y)-\cdots+(-1)^n(1-z)^{n-1}S_n(x)S_n(y)$$

and $S_k(x)$ is the kth elementary symmetric function of $x_1, \cdots, x_n$. In particular the generating function for pairs of sequences with RR forbidden is

$$\{1-S_1(x)S_1(y)+S_2(x)S_2(y)-\cdots+(-1)^nS_n(x)S_n(y)\}^{-1}. \tag{8}$$

Let $M^{(n)}(r; e,f)$ denote the number of pairs σ, τ of specification $[e]$, $[f]$ with exactly $N-r-1$ occurrences of LL and put

$$M^{(n)}(x, y, z)=\sum_{e,f,r}M^{(n)}(r; e,f)x^{e}y^{f}.$$

THEOREM 6. *We have*

$$(9)\qquad M^{(n)}(x, y, z) = \frac{(1-z)\left\{1 + \sum_{i,j=1}^{n} \frac{(1-z)x_iy_j}{1-(1-z)x_iy_j}\right\}}{1 - z\left\{1 + \sum_{i,j=1}^{n} \frac{(1-z)x_iy_j}{1-(1-z)x_iy_j}\right\}}.$$

In particular the generating function for pairs with LL forbidden is

$$(10)\qquad \left\{1 - \sum_{i,j=1}^{n} \frac{x_iy_j}{1+x_iy_j}\right\}^{-1}.$$

Let A, B denote any disjoint partition $A \neq \varnothing$, $B \neq \varnothing$ of the set (6). Let $C(\boldsymbol{e}, \boldsymbol{f}, k)$ denote the number of pairs of sequences σ, τ with exactly k B's. Put

$$F_k(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\boldsymbol{e},\boldsymbol{f}} C(\boldsymbol{e}, \boldsymbol{f}, k)\boldsymbol{x}^{\mathbf{e}}\boldsymbol{y}^{\mathbf{f}} \qquad (k = 0, 1, 2, \cdots),$$

where $C(\boldsymbol{e}, \boldsymbol{f}, 0) = 1$ $(N=0, 1)$; $F(\boldsymbol{x}, \boldsymbol{y}, z) = \sum_{k=0}^{\infty} z^k F_k(\boldsymbol{x}, \boldsymbol{y})$.

THEOREM 7. *We have*

$$(11)\quad F(\boldsymbol{x}, \boldsymbol{y}, z) = \frac{(1-z)F_0((1-z)x, y)}{1 - zF_0((1-z)x, y)} = \frac{(1-z)F_0(x, (1-z)y)}{1 - zF_0(x, (1-z)y)}.$$

THEOREM 8. *Let $C_A(\boldsymbol{e}, \boldsymbol{f})$ denote the number of pairs σ, τ with A forbidden. Then*

$$(12)\qquad \sum_{\boldsymbol{e},\boldsymbol{f}} C_A(\boldsymbol{e}, \boldsymbol{f})\boldsymbol{x}^{\mathbf{e}}\boldsymbol{y}^{\mathbf{f}} = \frac{1}{F_0(-\boldsymbol{x}, \boldsymbol{y})} = \frac{1}{F_0(\boldsymbol{x}, -\boldsymbol{y})}.$$

Hence

$$(13)\qquad F_A(\boldsymbol{x}, \boldsymbol{y})F_B(-\boldsymbol{x}, \boldsymbol{y}) = F_A(\boldsymbol{x}, \boldsymbol{y})F_B(\boldsymbol{x}, -\boldsymbol{y}) = 1,$$

where $F_A(\boldsymbol{x}, \boldsymbol{y})$ denotes the left member of (12).

A fuller account of these and other results will appear elsewhere.

REFERENCES

1. L. Carlitz, *Enumeration of sequences by rises and falls: A refinement of the Simon Newcomb problem*, Duke Math. J. **39** (1972), 267–280. MR **45** #1778.

2. ———, *Enumeration of up-down sequences*, Discrete Math. **4** (1973), 273–286. MR **47** #1625.

3. L. Carlitz and Richard Scoville, *Up-down sequences*, Duke Math. J. **39** (1972), 583–598. MR **47** #1627.

4. L. Carlitz, Richard Scoville and Theresa Vaughan, *Enumeration of permutations and sequences with restrictions*, Duke Math. J. **40** (1973), 723–741.

5. J. F. Dillon and D. P. Roselle, *Simon Newcomb's problem*, SIAM J. Appl. Math. **17** (1969), 1086–1093. MR **41** #1553.

Department of Mathematics, Duke University, Durham, North Carolina 27706

Current address (Vaughan): Department of Mathematics, University of North Carolina, Greensboro, North Carolina 27412