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HIGHER DIFFERENTIAL ALGEBRAS OF DISCRETE VALUATION RINGS

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Let S be a commutative ring with identity. Let T be a commutative S-algebra. $A_S(T)$ denotes the higher differential algebra over S, in the sense of Berger [1] or Kawahara-Yokoyama [5], with an index set consisting of all nonnegative integers. $\{d^n_{T/S}\}_{n=0,1,2,\cdots}$ denotes the canonical higher S-derivation of T into $A_S(T)$. In case S is the ring of all integers, we use simplified notations $A(T)$ and d^n_T $(n=0, 1, 2, \cdots)$.

Let R denote a complete discrete valuation ring, of a valuation v of unequal characteristic with maximal ideal $\mathfrak{m}=(\pi)$. Assume that the characteristic of $k=R/\mathfrak{m}$ is p. Let P be a coefficient ring of R. Let $\{\bar{c}_\iota\}_{\iota\in\Gamma}$ be a p-independent base of k and let c_ι be a representative of $\bar{c}_\iota$ chosen from P for every $\iota\in\Gamma$. The symbol $\wedge$ means the p-adic completion of P-algebra. By arguments developed by Berger or Kawahara-Yokoyama in the cited papers, and formal smoothness and flatness of P over the prime local ring, we can deduce the following theorem.

THEOREM 1. *$A(P)^\wedge = P[d^n_P c_\iota]^\wedge_{\iota\in\Gamma;n=0,1,2,\cdots}$, where $P[d^n_P c_\iota]_{\iota\in\Gamma;n=0,1,2,\cdots}$ is a polynomial ring over P in distinct indeterminates $d^n_P c_\iota$'s.*

For simplicity, we denote canonical images of $d^n_R c_\iota$ in $A(R)^\wedge$ by the same notation $d^n_R c_\iota$, for $\iota\in\Gamma$. Let $\{d^n\}_{n=0,1,2,\cdots}$ be the canonical higher derivation of the polynomial ring $P[X]$ into $(R\otimes_{P[X]} A(P[X]))^\wedge$. Let $f(X)$ be the Eisenstein polynomial over P such that $f(\pi)=0$. Then we have the following formula for every $n\geqq 1$.

$$d^n f(X) = f'(\pi)\, d^n X + \sum_{j\geqq 2} \frac{f^{(j)}(\pi)}{j!} \sum d^{i_1}X \cdots d^{i_j}X \tag{1}$$
$$+\, pG_n(d^1X, d^2X, \cdots ; \cdots , d^i c_\iota, \cdots),$$

where the second sum of the middle term is taken for the sets of integers

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$i_1, \cdots, i_j$ such that $i_1, \cdots, i_j \geqq 1$ and $i_1 + \cdots + i_j = n$, and

$$G_n(Z_1, Z_2, \cdots; \cdots, W_{\iota,i}, \cdots)$$

is a linear combination over R of countable monomials in indeterminates $\{Z_k, W_{\iota,i}\}_{\iota\in\Gamma;k,i=1,2,\ldots}$ of weight n, when we define as weight Z_i=weight $W_{\iota,i}=i$, such that for a given integer $t>0$ all but a finite number of coefficients of G_n belong to $\mathfrak{m}^t$ and G_n has no terms consisting of monomials of only Z_k's.

Solving equations $d^1f(X)=0$, $d^2f(X)=0, \cdots$ successively, we obtain relations

$$(2) \qquad (f'(\pi))^{2n-1}\, d_R^n \pi = F_n(\cdots, d_R^i c_\iota, \cdots),$$

where $F_n(\cdots, W_{\iota,i}, \cdots)$ have properties similar to G_n.

As an extended notion of Neggers' number for derivations of order 1 in Neggers [**6**] and Suzuki [**7**], [**8**], we give

DEFINITION. $\Delta_P^n(\pi) = \min v(\text{coefficient of } F_n) - (2n-1)v(f'(\pi))$ is called the nth Neggers number for (P, π), $n=1, 2, \cdots$.

We can show that $\Delta_P^n(\pi)$ is independent of the choice of $\{c_\iota\}_{\iota\in\Gamma}$.

Henceforth for a higher derivation $\{\partial^n\}_{n=0,1,2,\ldots}$ of P into P, R into R or k into k, we always assume that ∂^0=the identity map.

THEOREM 2. *The following four conditions are equivalent.*

(i) *Every higher derivation of P into P is extended to a higher derivation of R into R.*

(ii) $\Delta_P^n(\pi) \geqq 0$ *for all* $n=1, 2, \cdots$.

(iii) $\Delta_P^n(\pi) \geqq 1$ *for all* $n=1, 2, \cdots$.

(iv) *Every higher derivation of k into k is induced by a higher derivation of R into R.*

OUTLINE OF THE PROOF. The crucial point of the proof of this theorem is to show that (ii) implies (iii). Assume that (ii) is true and (iii) is false. Let n be the least integer such that $\Delta_P^n(\pi)=0$. Let e be the degree of $f(X)$. We can show that there exists a higher derivation $\{\partial^i\}_{i=0,1,2,\ldots}$ of R into R such that $\partial^n \pi \in \mathfrak{m}$. On the other hand, by the expansion of $(\partial^{ne} f)(\pi)$ according to (1) we can show that $\partial^n \pi \in \mathfrak{m}$, which is a contradiction.

If k is perfect or if R is tamely ramified, R satisfies conditions in Theorem 2.

THEOREM 3. *If R satisfies conditions in Theorem* 2, *the ideals* $(f'(\pi)$, $(f''(\pi)/2!), \cdots, (f^{(n)}(\pi)/n!))$ *are independent of the choice of P and π for all* $n=1, 2, \cdots$.

OUTLINE OF THE PROOF. Let ψ be an isomorphism of P onto another coefficient ring P' which induces an identity map on k. Expressing ψ

as a sum of all components of a higher derivation as in Heerema [**3**], it can be shown that ψ is extended to an automorphism λ of R by (i). ψ is extended to an isomorphism of $A_P(R)$ onto $A_{P'}(R)$. $A_P(R)$ and $A_{P'}(R)$ being graded algebras, we compare Fitting ideals of R-submodules of grade n of both algebras and deduce our theorem.

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