

THE METHOD OF EXTREMAL LENGTH

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Extremal length has become a useful tool in a wide variety of areas. The roots of the method can be traced back to the length-area comparisons in L. Ahlfors [1] and S. Warschawski [3], to the strip method of H. Grötzsch [1]–[15], and to even earlier works (see J. Jenkins [4, p. 7ff.] for a more complete historical background). In the 1940's Ahlfors and Beurling refined² those methods in a profound way; extremal length was introduced as a conformally invariant measure of curve families. This development appeared in Ahlfors-Beurling [4].

Subsequent applications of extremal length have been numerous and varied. There are geometric applications, function-theoretic applications, and applications which relate function-theoretic properties to geometric ones. In addition, there is a characteristically computational aspect which arises in connection with certain classical problems on univalent functions. I wish to discuss examples in each of these areas. There are many important topics I shall omit (prime ends, quasiconformal mapping, generalized modulus, and generalized capacity to mention a few); the extended bibliography will give a more complete picture.

1. The concept of extremal length. Let R be a surface (a surface, here, will be required to be connected, orientable, and to satisfy the second axiom of countability) with Riemannian metric ds_0 . An important special case will be a region R in the plane endowed with the Euclidean metric.

A second metric ds on R is said to be *conformally equivalent* to ds_0 if these two metrics give rise to the same angular measure in each tangent space of R . This condition is also expressed by the requirement that at each point of R the metrics are proportional: $ds = \rho ds_0$ where ρ is a positive function on R .

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² For a glimpse of this development through the eyes of one of the principals, see the recent book L. Ahlfors [9, pp. 50, 81].

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We shall define extremal length in a manner which emphasizes the conformal invariance. Let Γ be a family of curves on R . For each metric ds on R , define the ds -length of Γ to be

$$(1) \quad L(\Gamma, ds) = \inf_{\gamma \in \Gamma} \int_{\gamma} ds.$$

The area of R in the metric ds will be denoted by $A(ds)$. Except for the regularity conditions that will be made explicit in a moment, the *extremal length* of the family Γ will be defined by

$$(2) \quad \lambda(\Gamma) = \sup(L^2(\Gamma, ds)/A(ds)),$$

where the supremum is taken over all metrics ds which are conformally equivalent to ds_0 .

It is clear from the definition that $\lambda(\Gamma)$ depends only on the angular measurements induced by the original metric ds_0 , and not on the metric itself. Thus the definition can be applied if R is any Riemann surface. In addition, we see immediately that the following theorem on conformal invariance is valid.

THEOREM 1 (CONFORMAL INVARIANCE OF EXTREMAL LENGTH). *Let f be a 1-1 conformal mapping of a Riemann surface R onto a Riemann surface S . Let Γ be a family of curves on R . Then $\lambda(\Gamma) = \lambda(f(\Gamma))$.*

For constructions and estimations, extremal length will be most flexible when the regularity conditions on ds are weakest. On the other hand, the integrals in (1) and $A(ds)$ impose some degree of regularity. A number of different conditions have been suggested in the literature; see M. Ohtsuka [9, p. 70ff.], for a survey. A very satisfactory requirement, and the one I shall adopt, is the following: the supremum in (2) is taken over all ds of the form $ds = \rho ds_0$, where ρ is a nonnegative Borel measurable function on R . This insures the existence of both integrals $\int_{\gamma} ds$ and $A(ds)$, although they might be infinite. In the definition (2) we also set $0/0 = \infty/\infty = 0$. It should be noted that ds is not necessarily a Riemannian metric in the usual sense—it may have zeros, and it need not be differentiable. Such a ds is called a *linear density*. The conformal invariance theorem, Theorem 1, still holds. In fact, we have (M. Ohtsuka [9, pp. 80–84]):

THEOREM 2. *Let $f: R \rightarrow S$ be a complex analytic mapping of Riemann surfaces. Let Γ be a curve family on R . Then $\lambda(\Gamma) \leq \lambda(f(\Gamma))$.*

{An alternate definition of extremal length given in L. Ahlfors [6] admits competing metrics ds which need not be conformally equivalent to ds_0 . If ds is any Riemannian metric on R , the ratio ds/ds_0 has a maximum λ_1 and a minimum λ_2 at each point of R . The dilatation K is defined as

$K = \lambda_1/\lambda_2$. It is proved that $\lambda(\Gamma) = \sup(L^2(\Gamma, ds)/A'(ds))$, where $A'(ds) = \iint_R K d\omega$ and $d\omega$ is the area element in the metric ds .

In certain simple cases extremal length can be computed explicitly. Consider a rectangle R in the plane, $R = \{(x, y) | 0 < x < a, 0 < y < b\}$. Let Γ consist of all arcs in R which join the sides of length b . A lower bound for $\lambda(\Gamma)$ is obtained by computing $L^2(\Gamma, ds)/A(ds)$ for any specific ds ; if we use the Euclidean metric $|dz|$ we obtain the lower bound

$$\lambda(\Gamma) \geq L^2(\Gamma, |dz|)/A(|dz|) = a^2/ab = a/b.$$

An upper bound can be found by choosing a subfamily of Γ . The horizontal segments, $\gamma_y: t \mapsto (t, y)$ for $0 < t < a$, yield

$$L^2(\Gamma, \rho |dz|) \leq \left[\int_{\gamma_y} \rho |dz| \right]^2 \leq \left[\int_{\gamma_y} \rho^2 |dz| \right] \left[\int_{\gamma_y} |dz| \right] = a \int_{\gamma_y} \rho^2 dx;$$

integrate for $y \in (0, b)$ and obtain

$$bL^2(\Gamma, \rho |dz|) \leq a \iint_R \rho^2 dx dy = aA(\rho |dz|).$$

Thus $L^2(\Gamma, \rho |dz|)/A(\rho |dz|) \leq a/b$, and since ρ was arbitrary $\lambda(\Gamma) \leq a/b$. This proves

THEOREM 3. *Let R be a rectangle of sides a and b . Let Γ be the family of arcs in R which join the sides of length b . Then $\lambda(\Gamma) = a/b$.*

Two other explicit values for extremal lengths ds are given in the following elementary theorem.

THEOREM 4. *Let R be the annulus $a < |z| < b$. Let Γ be the family of arcs in R which join the two contours. Let Γ^* be the family of closed curves in R which separate the two contours. Then $\lambda(\Gamma) = \lambda^{-1}(\Gamma^*) = (1/2\pi)\log(b/a)$.*

2. Geometric inequalities. There are a number of purely geometric applications of extremal length. The simplest example concerns the general quadrilateral (a Jordan curve with four distinguished points).

THEOREM 5. *Let R be a general quadrilateral of area A . Let a be the length of the shortest arc in R connecting one pair of opposite sides. Let b be the length of the shortest arc in R connecting the other pair of sides. Then $a \cdot b \leq A$.*

A purely geometric proof of this theorem (E. Rengel [1], O. Teichmüller [1]) was given by A. Besicovitch [1] in response to a function-theoretic proof presented to him by Charles Löwner. The geometric proof is difficult. The use of extremal length, however, makes the proof trivial.

Let Γ be the family of arcs in R which join one pair of opposite sides, and let Γ^* be the family of arcs in R which join the other two sides. There is a conformal map f of R onto some rectangle such that $f(\Gamma)$ and $f(\Gamma^*)$ are the families of arcs which join opposite sides of the rectangle. By Theorem 3, $\lambda(f(\Gamma)) \cdot \lambda(f(\Gamma^*)) = 1$. By the conformal invariance, $\lambda(\Gamma) \cdot \lambda(\Gamma^*) = 1$. The Euclidean metric $|dz|$ in R can be used to give lower bounds for these latter two extremal lengths. Thus

$$\frac{a^2}{A} \cdot \frac{b^2}{A} = \frac{L^2(\Gamma, |dz|)}{A(|dz|)} \cdot \frac{L^2(\Gamma^*, |dz|)}{A(|dz|)} \leq \lambda(\Gamma) \cdot \lambda(\Gamma^*) = 1,$$

and the theorem follows at once.

There is a generalization of Theorem 5 to higher connectivity (Rodin [7]). Let E be a compact set contained in the interior of the quadrilateral R , and suppose that $R_1 = R - E$ is connected. Theorem 5 would be false if we replaced R by R_1 . It is possible to generalize from R to R_1 if the definitions of a and b are suitably generalized. Let Γ_1 be the family of arcs in R_1 which join one pair of opposite sides. Let Γ_1^* be the family of curves (a *curve* is to mean a union of arcs) in R_1 which separate that pair of sides. An extremal length theorem asserts that $\lambda(\Gamma_1) \cdot \lambda(\Gamma_1^*) = 1$. It follows immediately that if a is the infimum of the lengths of Γ_1 , and if b is the infimum of the lengths of Γ_1^* , then $a \cdot b \leq A$, where A is the area of R_1 .

The extremal lengths $\lambda(\Gamma_1)$, $\lambda(\Gamma_1^*)$ considered above play an important role in characterizing certain function-theoretic properties of the set E . We shall return to such applications in §5.

Here is another geometric application of extremal length (O. Teichmüller, C. Löwner, P. M. Pu [1], L. Keen [1], B. Rodin [7]):

THEOREM 6. *Let R be a torus with a Riemannian metric. Let a be the length of the shortest nontrivial cycle on R , and let A be the area of R . Then $a^2 \leq 2A/\sqrt{3}$.*

This theorem can be proved easily by first mapping R conformally onto a flat torus represented by a parallelogram $0, 1, \tau, 1 + \tau$ in the plane. One can always choose τ to be in a fundamental region of the modular group, so we may assume $-\frac{1}{2} \leq \operatorname{Re} \tau \leq \frac{1}{2}$, $|\tau| \geq 1$, $\operatorname{Im} \tau > 0$. Thus $\operatorname{Im} \tau \geq \sqrt{3}/2$. In the parallelogram, the extremal length of the horizontal line segments is easily seen to be $1/\operatorname{Im} \tau$. Therefore the corresponding cycles Γ on the torus have extremal length $1/\operatorname{Im} \tau$. Therefore, if ds_0 is the original metric on the torus then $L^2(\Gamma, ds_0)/A(ds_0) \leq 2/\sqrt{3}$ for these particular cycles Γ , and the theorem follows.

The following theorem was proved by C. Blatter [1]. It is a profound generalization of Theorem 6, although the constant is no longer best possible.

THEOREM 7. *Let R be a compact surface of genus g with a Riemannian metric. Let a be the length of the shortest nontrivial cycle on R . Let A be the area of R . Then*

$$a^2 < (2/\pi)((g+1)!)^{1/g} A.$$

The proof of Theorem 7 is too long to repeat here. However, it will be useful to sketch the main ideas in order to point out an important extremal length result which arose in the course of the proof.

There is a linear mapping (de Rham, Hodge) of the 1-cycles on R to the harmonic 1-forms on R . The harmonic differential σ_c which corresponds to the 1-cycle c under this mapping satisfies the two properties:

$$(3) \quad \int_c \omega = (\omega, \sigma_c) \quad \text{for all closed 1-forms } \omega,$$

$$(4) \quad \int_d * \sigma_c = c \times d \quad \text{for all 1-cycles } d,$$

where $(\omega, \sigma_c) = \iint_R \omega \wedge * \sigma_c$ is the inner product, and $c \times d$ is the intersection number.

The norm $(\sigma_c, \sigma_c)^{1/2}$ can be related to extremal length. Let $\lambda([c])$ denote the extremal length of the homology class of c . C. Blatter [1] and R. Accola [1] proved independently that

$$(5) \quad \lambda([c]) = (\sigma_c, \sigma_c).$$

The important result (5) (it will be discussed in more detail later) shows that once a homology basis c_1, c_2, \dots, c_{2g} is chosen for the 1-cycles, the $2g \times 2g$ matrix $S = \|(\sigma_{c_i}, \sigma_{c_j})\|$ determines the extremal lengths $\lambda([c])$ of all homology classes $[c]$ on R by means of

$$(6) \quad \lambda([\sum n_i c_i]) = \sum n_i n_j (\sigma_{c_i}, \sigma_{c_j}) = n S n^t.$$

There is a theorem of Minkowski which guarantees, for any positive definite $2g \times 2g$ matrix Q of determinant 1, the existence of a nonzero integer vector m which satisfies $m Q m^t \leq (2/\pi)[(g+1)!]^{1/g}$. The next part of Blatter's proof consists in showing that the matrix S satisfies the hypotheses of Minkowski's theorem if the homology basis c_1, c_2, \dots, c_{2g} is suitably chosen. After that is accomplished the vector $m = (m_1, m_2, \dots, m_{2g})$ can be used to form a nontrivial $c = m_1 c_1 + m_2 c_2 + \dots + m_{2g} c_{2g}$ with $\lambda([c]) \leq (2/\pi)[(g+1)!]^{1/g}$. Therefore, if ds_0 is the original metric on R then $L^2([c], ds_0)/A(ds_0) \leq (2/\pi)[(g+1)!]^{1/g}$ and the theorem follows.

The Accola-Blatter result (5) has interesting ramifications. For example, it shows that the matrix S is completely determined by the extremal lengths of a finite number of homology classes. The matrix S , in turn, determines

the Riemann period matrix of the compact Riemann surface underlying R . Hence, by Torelli's theorem, we conclude that a finite number of extremal lengths of homology classes provide a complete set of conformal invariants for compact Riemann surfaces. The same is true for plane regions of finite connectivity.

Other ramifications of (5) involve noncompact Riemann surfaces. Accola [1] proved (5) for an arbitrary R , compact or not. A. Marden [1] and Rodin [1] considered other kinds of homology classes. Let $[c]_1$ denote the homology classes of c modulo dividing cycles. Let $[c]_2$ denote the relative homology class of c (chains may have infinitely many simplexes). Marden [1] proved that $\lambda([c]_2) = (\sigma_{h_0}, \sigma_{h_0})$; Rodin [1] proved that $\lambda([c]_1) = (\sigma_{h_{se}}, \sigma_{h_{se}})$. Here σ_{h_0} and $\sigma_{h_{se}}$ are the projections of σ_c onto the subspaces Γ_{h_0} and $\Gamma_{h_{se}}$ of the Hilbert space Γ_h of square integrable 1-forms on R . The subspace Γ_{h_0} is the orthogonal complement of $\Gamma_{h_e}^*$, where Γ_{h_e} is the space of exact harmonic 1-forms. The subspace $\Gamma_{h_{se}}$ consists of the semixact harmonic 1-forms.

Another generalization of (5) was recently obtained by C. D. Minda [2]. He was able to relate the norm (σ_c, σ_c) to extremal length in the case that c is an arbitrary 1-chain, not necessarily a cycle. This result is especially interesting if R is a compact Riemann surface, for it leads to an extremal length characterization of principal divisors—a geometric counterpart to the classical theorem of Abel.

Extremal length has been used to obtain length-area inequalities related to flows. Let c be a 1-cycle on a compact Riemannian 2-space R . Then, in the sense of F. Klein [1], there is a steady fluid flow on R corresponding to c ; namely, the flow which has the property that along any closed curve d the change in the potential is the intersection number $c \times d$. Let a be the length of the shortest geodesic homologous to c . Let b be the length of the shortest geodesic, not necessarily a cycle, along which the potential drop is 1. Then $a \cdot b \leq A$, where A is the area of R (Rodin [7]).

3. Infinite extremal length. Having discussed some geometric applications of extremal length, we turn next to applications in the area of analysis. Just as measure theory provides the notion of a negligible point set (measure zero), extremal length provides the notion of a negligible curve family. Following B. Fuglede [1], we say that a property holds for *almost all curves* in a curve family Γ if the subfamily of exceptional curves has infinite extremal length. (In this context the reciprocal of the extremal length, called the *modulus*, may be more natural.) This notion may be justified somewhat by the following properties of infinite extremal length: For any curve families Γ , Γ_0 , Γ_1, \dots

$$(7) \quad \lambda(\Gamma_0) = \infty \Rightarrow \lambda(\Gamma \cup \Gamma_0) = \lambda(\Gamma)$$

and

$$(8) \quad \lambda(\Gamma_n) = \infty \quad \text{for } n = 1, 2, \dots \Rightarrow \lambda(\cup \Gamma_n) = \infty.$$

Some examples may help to illustrate the property of infinite extremal length. Consider the horizontal line segments $\gamma_y = \{(x, y) | 0 \leq x \leq 1\}$, and let $\Gamma = \{\gamma_y | y \in E\}$ where E is a measurable set of real numbers. Then $\lambda(\Gamma) = \infty$ if and only if E has measure zero.

As a second example, let E be a compact point set in $|z| < 1$, and let $\bar{\Gamma}$ consist of all arcs which join $|z| = 1$ to E . Then $\lambda(\bar{\Gamma}) = \infty$ if and only if the logarithmic capacity of E vanishes.

There are many function-theoretic applications of this notion of infinite extremal length. A striking example of its usefulness occurs in the problem of the existence of boundary values for functions with a finite Dirichlet integral. For simplicity, consider the case of a C^1 -function u defined in a plane region R . The Dirichlet integral of u is $D(u) = \iint_R (u_x^2 + u_y^2) dx dy$. We say that an arc $\gamma: [0, 1] \rightarrow R$ in R tends to the boundary of R if $\bigcap \{Cl \gamma[(t, 1)] | 0 \leq t < 1\} \subset \partial R$. The next theorem (M. Ohtsuka [8], [9]) shows that, in a general sense, u has boundary values if $D(u) < \infty$. The significant point is that no assumptions whatever are required concerning the geometric boundary of R .

THEOREM 8. *If $D(u) < \infty$ then u has a limit along almost all arcs in R which tend to the boundary.*

An extremal length argument makes the proof trivial. Consider the linear density $ds = |\text{grad } u| |dz|$. Then $A(ds) = D(u) < \infty$. Let Γ be the family of arcs in R which tend to the boundary. The subfamily $\Gamma' = \{\gamma \in \Gamma | \int_\gamma ds = \infty\}$ clearly has infinite extremal length. If $\gamma \in \Gamma - \Gamma'$ then $\int_\gamma ds < \infty$, and hence $\int_\gamma |du| < \infty$. Thus u has a limit along each such γ , and the theorem is proved.

In the situation of Theorem 8, suppose u has a limit along two arcs which tend to the same boundary point of R . The two limiting values would not, in general, be expected to agree; for example, the arcs might tend to different prime ends. The next result (Ohtsuka [9]) gives a useful condition for equality of these limits.

THEOREM 9. *Let c_1 and c_2 be arcs with a common terminal point z_0 , and suppose they form the boundary of a Jordan region Ω . Let u be continuous in $Cl \Omega - \{z_0\}$ and satisfy $D(u) < \infty$. If u tends to a limit a along c_1 , and to a limit b along c_2 , then $a = b$.*

Again, extremal length provides a simple method of proof. Consider the family of circular cross cuts of Ω with center at z_0 and radius $\leq \varepsilon$. It is elementary that $\lambda(\Gamma) = 0$. Therefore the density $ds = |\text{grad } u| |dz|$ satisfies $\inf_{\gamma \in \Gamma} \int_\gamma ds = 0$, and hence $\inf_{\gamma \in \Gamma} \int |du| = 0$. It follows that $a = b$.

An interesting consequence of Theorem 9 is that the boundary values of a Dirichlet finite function cannot have a jump discontinuity if the region is bounded by Jordan curves. Other types of discontinuities are possible. Further results on the boundary values of Dirichlet finite functions, though not using the method of extremal length, are contained in J. Doob [1] and J. L. Lions and E. Magenes [1].

M. Ohtsuka [8], [9] uses extremal length to develop a very general form of Dirichlet's principle. His result, Theorem 10 below, shows that this notion of *almost all curves* describes perfectly the mode in which a harmonic function can be made to imitate the boundary behavior of a Dirichlet finite function.

THEOREM 10. *Let u be a C^1 -function on a hyperbolic Riemann surface R with $D(u) < \infty$. Then there is a unique harmonic function U such that $U - u$ has limit zero along almost all arcs in R which tend to the boundary.*

The notion of almost all curves has interesting applications to extremal length itself. This will be discussed next.

4. Extremal metrics. Suppose Γ is a curve family on a Riemann surface R , and $0 < \lambda(\Gamma) < \infty$. We are interested in the problem of existence and uniqueness for an *extremal metric*, that is, a metric ds which satisfies $\lambda(\Gamma) = L^2(\Gamma, ds)/A(ds)$. Simple examples show that an extremal metric need not exist in general. In the following discussion Theorems 11, 12, and 13 are based on K. Strebel [2] and N. Sita [3].

Consider the set $P(\Gamma)$ consisting of all linear densities ds on R which satisfy $A(ds) < \infty$ and $L(\Gamma, ds) \geq 1$. Thus $\lambda^{-1}(\Gamma) = \inf\{A(ds) \mid ds \in P(\Gamma)\}$. Note that $P(\Gamma)$ is a convex subset of a Hilbert space³ where $\sqrt{A(ds)}$ is the norm. Therefore $\lambda^{-1}(\Gamma) = A(ds_0)$ for the unique linear density ds_0 of minimum norm in the closure $\text{Cl } P(\Gamma)$ of $P(\Gamma)$. The following theorem characterizes the linear densities in $\text{Cl } P(\Gamma)$.

THEOREM 11. *$\text{Cl } P(\Gamma)$ consists of all nonnegative linear densities ds such that $\int_\gamma ds \geq 1$ for almost all $\gamma \in \Gamma$.*

We can now answer the question about extremal metrics.

THEOREM 12. *Let $0 < \lambda(\Gamma) < \infty$. There is a linear density ds_0 which is the extremal metric for a subfamily $\Gamma_0 \subset \Gamma$. Γ_0 satisfies $\lambda(\Gamma - \Gamma_0) = \infty$, and hence $\lambda(\Gamma_0) = \lambda(\Gamma)$. The family Γ determines ds_0 uniquely (a.e.) up to a positive multiplicative constant.*

³ The Hilbert space consists of square integrable densities $\rho|dz|$ where the functions ρ are no longer required to be nonnegative.

Let us call the metric ds_0 of Theorem 12 the *generalized extremal metric* for Γ . It satisfies the following inequality, a consequence of the "law of cosines":

THEOREM 13. *If $ds \in \text{Cl } P(\Gamma)$, and if ds_0 is the generalized extremal metric for Γ , then $A(ds_0) \leq A(ds) - A(ds - ds_0)$.*

Next, I wish to present an explicit method for finding the extremal metric. The method is restricted to one-parameter curve families. However, it has much wider applicability because it can often be applied to one-parameter subfamilies of a general curve family (see §6 below).

For each $s \in (a, b)$, suppose $c_s: t \rightarrow c_s(t)$ is a curve in R . The domain of c_s may depend on s ; suppose it is an open interval $t_0(s) < t < t_1(s)$. We shall assume that R is a region in the x, y -plane, and that the mapping $(s, t) \rightarrow c_s(t) = (x, y)$ is 1-1 and has positive Jacobian determinant $\partial(x, y)/\partial(s, t)$. Then the extremal length of the curve family $\Gamma = \{c_s\}_{a < s < b}$ can be calculated exactly:

THEOREM 14. *Let*

$$(9) \quad l(s) = \int_{t_0(s)}^{t_1(s)} \frac{|dc_s(t)/dt|^2}{\partial(x, y)/\partial(s, t)} dt.$$

Then

$$(10) \quad \frac{1}{\lambda(\Gamma)} = \int_a^b \frac{ds}{l(s)}.$$

Theorem 14 can be considered as a generalization of the known formulas for calculating the extremal length of a family of parallel line segments, concentric circular arcs, and line segments on a pencil (cf. Theorems 2.4 and 2.6 in M. Ohtsuka [9]). Integrals of the form (10) appeared in Ahlfors [1] and Warschawski [1]. I have not determined the weakest regularity conditions which would justify the following computations.

For the proof of Theorem 14, define

$$(11) \quad \rho_0(c_s(t)) = \frac{1}{l(s)} \cdot \frac{|dc_s(t)/dt|}{\partial(x, y)/\partial(s, t)}$$

where $l(s)$ is given by (9). We shall show that $\rho_0(z)|dz|$ is the extremal metric for Γ . We first note that $\int_{c_s} \rho_0|dz| = 1$ for each c_s :

$$\begin{aligned} \int_{c_s} \rho_0(z) |dz| &= \int_{t_0(s)}^{t_1(s)} \rho_0(c_s(t)) \left| \frac{d}{dt} c_s(t) \right| dt \\ &= \int_{t_0(s)}^{t_1(s)} \frac{1}{l(s)} \cdot \frac{|dc_s(t)/dt|^2}{\partial(x, y)/\partial(s, t)} dt = \frac{1}{l(s)} \cdot l(s) = 1. \end{aligned}$$

Next, suppose that $\rho(z)|dz|$ is any linear density which satisfies $\int_{c_s} \rho(z)|dz| \geq 1$ for each c_s . Then for each s

$$\int_{c_s} [\rho(z) - \rho_0(z)] |dz| \geq 0,$$

$$\int_{t_0(s)}^{t_1(s)} [\rho(c_s(t)) - \rho_0(c_s(t))] \left| \frac{d}{dt} c_s(t) \right| dt \geq 0,$$

and so

$$(12) \quad \int_{t_0(s)}^{t_1(s)} \frac{1}{l(s)} [\rho(c_s(t)) - \rho_0(c_s(t))] \left| \frac{d}{dt} c_s(t) \right| dt \geq 0.$$

We integrate (12) for $s \in (a, b)$ and write the resulting inequality in the form

$$\iint_S [\rho(c_s(t)) - \rho_0(c_s(t))] \rho_0(c_s(t)) \frac{\partial(x, y)}{\partial(s, t)} ds dt \geq 0,$$

which gives $\iint_R \rho(z) \rho_0(z) dx dy \geq \iint_R \rho_0^2(z) dx dy$. By use of the Schwarz inequality we then arrive at

$$(13) \quad \iint_R \rho^2 dx dy \geq \iint_R \rho_0^2 dx dy.$$

Equation (13) shows that $\rho_0|dz|$ is the extremal metric. It can be used to calculate the extremal length:

$$\begin{aligned} \frac{1}{\lambda(\Gamma)} &= \iint_R \rho_0^2 dx dy = \iint_S \rho_0^2(c_s(t)) \frac{\partial(x, y)}{\partial(s, t)} ds dt \\ &= \int_a^b \int_{t_0(s)}^{t_1(s)} \frac{|dc_s(t)/dt|^2}{\partial(x, y)/\partial(s, t)} dt \frac{ds}{l^2(s)} = \int_a^b \frac{ds}{l^2(s)} = \int_a^b \frac{ds}{l(s)}, \end{aligned}$$

and we arrive at (10). As a corollary of the proof we obtain

THEOREM 15. *In the situation of Theorem 14, the extremal metric for Γ is*

$$\rho_0(z) |dz| = \frac{1}{l(s)} \frac{|dc_s(t)/dt|}{\partial(x, y)/\partial(s, t)} |dz|$$

where $z = c_s(t)$.

5. Extremal distance and function theoretic null-sets. Extremal length provides a link for relating geometric and function-theoretic quantities. This aspect can be illustrated by the several notions of extremal distance.

Let s_1 and s_2 be disjoint subsets contained either in a Riemann surface R or on its border. Let Γ be the family of arcs in R which join s_1 and s_2 . The extremal length of Γ is then called the *extremal distance* between s_1 and s_2 , and it is denoted by $\lambda(s_1, s_2)$.

We can modify this definition by considering other ways in which a curve might be said to "join" s_1 and s_2 . For example, consider the Kerékjártó-Stoïlow compactification R_K of R ; R_K is obtained by adding to R an ideal point for each boundary component of R . An arc on R_K determines a curve on R . If an arc on R_K joins s_1 to s_2 , the corresponding curve on R might be said to join s_1 to s_2 "via the boundary components". Let Γ_K consist of all arcs on R_K which join s_1 to s_2 , and let $\lambda_K(s_1, s_2)$ be the extremal length of the curve family on R determined by Γ_K . Then $\lambda_K(s_1, s_2)$ is another notion of extremal distance between s_1 and s_2 .

A third notion of extremal distance is obtained by considering the one-point compactification R_∞ of R . An arc on R_∞ which joins s_1 to s_2 determines a curve on R which joins s_1 to s_2 "via the ideal boundary point". Let Γ_∞ consist of all arcs on R_∞ which join s_1 to s_2 , and let $\lambda_\infty(s_1, s_2)$ denote the extremal length of the corresponding curve family on R .

Note that $\Gamma \subset \Gamma_K \subset \Gamma_\infty$ when we consider these as curve families on R . Therefore the three notions of extremal distance satisfy

$$(14) \quad \lambda(s_1, s_2) \geq \lambda_K(s_1, s_2) \geq \lambda_\infty(s_1, s_2).$$

The conditions for equality in (14) are related to function theoretic null-sets:

THEOREM 16. *Let R be a plane region with complement E . The logarithmic capacity of E vanishes if and only if $\lambda(s_1, s_2) = \lambda_\infty(s_1, s_2)$ for every pair of disjoint closed disks s_1, s_2 contained in R .*

THEOREM 17. *Let R be a plane region with complement E . The set E is a removable singularity for every Dirichlet finite analytic function if and only if $\lambda(s_1, s_2) = \lambda_K(s_1, s_2)$ for every pair of disjoint closed disks s_1, s_2 contained in R .*

Theorems 16 and 17 can be generalized to Riemann surfaces merely by replacing the conditions relating to E by the intrinsic conditions $R \in O_{HD}$ in Theorem 16, and $R \in O_{KD}$ in Theorem 17. Theorem 17 was proved in Rodin [4], and Theorem 16 was proved in Minda [1]. The earliest result of this type is due to Ahlfors-Beurling [4]; an extension to higher dimensions is given in J. Väisälä [3].

6. The classical origins. It has already been mentioned that extremal length had its roots in the techniques used over forty years ago to investigate certain problems in complex analysis. It is of interest to see if the modern theory of extremal length can be profitably reapplied to those

original problems. A program of this sort was recently begun by J. Jenkins-K. Oikawa [6] (cf. also D. Gaier [1] and A. Obrock [1]). In addition to improving a result of Hayman, Jenkins and Oikawa showed that extremal length methods provide significant simplifications in the proofs of the fundamental inequalities of Ahlfors.

One of the original problems was the behavior of the Riemann mapping function near a boundary point of the domain. Ahlfors introduced the useful normalization of applying a logarithmic transformation to the domain and range. The result is a conformal mapping $w=f(z)$ of a strip domain S onto a parallel strip. As z tends to a certain infinite boundary element, $f(z)$ tends to $+\infty$.

Suppose that the strip domain S is determined by the graphs of two real valued functions $y=\varphi_-(x)$ and $y=\varphi_+(x)$:

$$S = \{z = x + iy \mid \varphi_-(x) < y < \varphi_+(x)\}.$$

The conformal mapping $w=f(z)$ sends S onto a parallel strip $R = \{w = u + iv \mid -\pi/2 < v < \pi/2\}$, and $f(z) \rightarrow +\infty$ as $z \rightarrow +\infty$, $z \in S$. There are a number of problems of interest in this situation. For example, what properties of φ_- and φ_+ guarantee the existence of $\lim_{z \rightarrow +\infty} [f(z) - z]$ (the problem of differentiability at the boundary)?

In their extremal length formulation, these kinds of questions lead to the problem of estimating extremal distance in a quadrilateral. Let $Q = Q(x_0, x_1)$ be the quadrilateral region $\{z = x + iy \mid z \in S, x_0 < x < x_1\}$. Let $M = M(x_0, x_1)$ be the extremal length of all arcs in Q which join the vertical sides of Q . The crucial problem is to obtain useful estimates of M .

Theorem 14 is useful for this purpose. It gives the exact extremal length of a one-parameter curve family. If Γ is any one-parameter family of arcs in Q which join the vertical sides, then $\lambda(\Gamma)$ serves as an upper bound for M . If Γ^* is a one-parameter family of arcs in Q which separate the vertical sides, then $\lambda^{-1}(\Gamma^*)$ serves as a lower bound for M . With judicious choices of Γ and Γ^* , useful bounds on M can be obtained. (The bounds will be exact if Γ and Γ^* correspond to the coordinate grid in the rectangular conformal image of Q .)

An obvious first choice for Γ is the family $\{c_s\}_{0 < s < 1}$, where $c_s(t) = t + i[s\varphi_-(t) + (1-s)\varphi_+(t)]$ for $x_1 < t < x_2$. An application of Theorem 14 yields

$$\begin{aligned} l(s) &= \int_{x_1}^{x_2} \frac{1 + [s\varphi'_-(t) + (1-s)\varphi'_+(t)]^2}{\varphi_+(t) - \varphi_-(t)} dt, \\ (15) \quad \lambda(\Gamma) &= \left\{ \int_0^1 \frac{ds}{l(s)} \right\}^{-1} \leq \int_0^1 l(s) ds \\ &= \int_{x_1}^{x_2} \frac{3 + \varphi'^2_+(t) + \varphi'_+(t)\varphi'_-(t) + \varphi'^2_-(t)}{3[\varphi_+(t) - \varphi_-(t)]} dt. \end{aligned}$$

Thus the integral in (15) is an upper bound for M . It is essentially the same as the one derived by Jenkins-Oikawa [6]; we have derived it in a different way.

An obvious first choice for Γ^* is the family $\{c_s\}$ of vertical line segments $c_s(t) = s + it$ ($\varphi_-(s) < t < \varphi_+(s)$). The calculation of $\lambda(\Gamma^*)$ yields

$$(16) \quad \frac{1}{\lambda(\Gamma^*)} = \int_{\varphi_1}^{\varphi_2} \frac{dt}{\varphi_+(t) - \varphi_-(t)}.$$

The integral in (16) is therefore a lower bound for M . It also is the same as the lower bound derived by Jenkins-Oikawa.

Substantial improvements in the bounds (15) and (16) can be obtained from more sophisticated choices of Γ and Γ^* . This work is currently being carried out by Professor Warschawski and myself. In the case of symmetric regions, the use of circular arcs orthogonal to the graphs of φ_- and φ_+ leads to a relatively simple derivation of the estimates in the remarkable, though difficult, paper of Gol'dberg-Stročik [1]. New estimates of wider applicability have been obtained by using other curve families. We are planning a joint publication on these topics in the near future.

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