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SMOOTH MAPS OF CONSTANT RANK

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1. **Introduction.** In this announcement the Smale-Hirsch classification of immersions ([**8**], [**5**]) is extended to maps of arbitrary constant rank, under certain conditions on the source manifold.

THEOREM 1. *If M is open and has a proper Morse function with no critical points of index $>k$, then the differential map $d:\mathrm{Hom}_k(M, W)\to \mathrm{Lin}_k(TM, TW)$ is a weak homotopy equivalence.*

(A manifold with such a Morse function will be said to have *geometric dimension* $\leqq k$. We will write geo dim $M\leqq K$.)

Notation. M and W are smooth manifolds with tangent bundles TM, TW; $\mathrm{Hom}_k(M, W)$ is the space of smooth maps of rank k from M to W, with the C^1-compact-open topology; $\mathrm{Lin}_k(TM, TW)$ is the space of continuous maps: $TM\to TW$ which are fiberwise linear maps of rank k, with the compact open topology; $d(f)=df$.

REMARKS. 1. Weakening the hypotheses leads to false statements. If M is not open there are counterexamples when $k=\dim W$ as in [**7**]. Otherwise, take M to be the parallelizable manifold $S^{k+1}\times R$; then the identity map of M can be covered by $H\in \mathrm{Lin}_k(TM, TM)$ but H cannot be homotopic to the differential of an $f\in \mathrm{Hom}_k(M, M)$ since such an f (by Sard's theorem) would be null-homotopic. I owe this example to David Frank.

2. When $k=\dim M$ this gives the Smale-Hirsch theorem for open manifolds, but when $k=\dim W$ this does not give the full classification of submersions [**7**]. The missing cases will be considered in a future article. (ADDED IN PROOF. A necessary and sufficient condition for $H\in \mathrm{Lin}_k(TM, TW)$ to be homotopic to the differential of some $f\in \mathrm{Hom}_k(M, W)$ is given, for arbitrary open M, in M. L. Gromov, *Singular smooth maps*, Mat. Zametki **14** (1973), 509–516. It is equivalent to requiring that H factor through a k-dimensional bundle over a k-dimensional complex.) Immersions and submersions are the only overlap between this theorem and Feit's classification of k-mersions (maps of rank everywhere$\geqq k$) [**2**]

3. This theorem is not a special case of Gromov's theorem [**3**], since

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having rank k is not an open condition in general. We will, however, make heavy use of the ideas and results of [**3**] throughout this work.

4. A map of constant rank is locally a submersion followed by an immersion, i.e., a *subimmersion.* The ensuing local "stability" is the key to our proof. It also follows that inverse images of points under a map of constant rank foliate the source manifold. An application of Theorem 1 is then this weak form of a theorem of [**4**]. *On an open manifold M of geometric dimension $\leqq k$, any plane field σ of codimension k is homotopic to an integrable field.* The proof is immediate, since projection onto the complementary bundle $\sigma^{\perp}$ can be considered as a bundle map of rank k from TM to the tangent bundle of the total space of $\sigma^{\perp}$.

The proof of Theorem 1 has two main steps: First, the manifold M is assumed to be highly coconnected; then the general case is reduced to this special one.

2. **Proof for highly coconnected manifolds.** Let $a(0)=a(1)=0$, $a(2)=a(3)=1$, and $a(x)=\frac{1}{2}(x-1)$ if $x\geqq 4$.

THEOREM 1′. *Let* $\dim M=n$. *If* $\text{geo}\dim M\leqq\min(a(n),k)$, *then* d: $\text{Hom}_k(M, W)\to\text{Lin}_k(TM, TW)$ *is a weak homotopy equivalence.*

The theorem proving machine ([**3**], [**6**]) reduces the proof to showing that the restriction map $\text{Hom}_k(V, W)\to\text{Hom}_k(U, W)$ has the covering homotopy property, when $U\subset V\subset M$ are n-dimensional submanifolds, and $V=U\cup\text{handle of index }\lambda\leqq\min(a(n),k)$. This is not true in general (see Figure 1), but, as pointed out to me by Edgar Feldman, the *weak* covering homotopy property [**1**] is sufficient (this allows a preliminary vertical homotopy; see below). Using 3.2.3 of [**3**] (r-microflexible implies r-flexible) we can further reduce our problem to the following lemma.

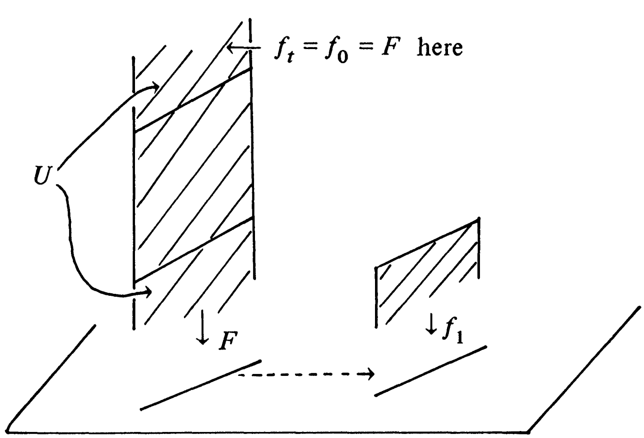

FIGURE 1

WEAK MICRO-COVERING HOMOTOPY LEMMA. *Suppose we are given* U, V *as above, a compact* P *and continuous maps* $F:P\to \mathrm{Hom}_k(V, W)$ *and* $f:P\times[0,1]\to \mathrm{Hom}_k(U, W)$ *with* $f_{p,0}=F_p|U$ *for* $p\in P$. *Then there exist* $\varepsilon>0$ *and a continuous* $\tilde{F}:P\times[-1,\varepsilon]\to \mathrm{Hom}_k(V, W)$ *with* $\tilde{F}_{p,-1}=F_p$ *for* $p\in P$, *such that* $\tilde{F}_{p,t}|U=f_{p,0}$ *if* $t\leqq 0$ *and* $=f_{p,t}$ *if* $0\leqq t\leqq\varepsilon$, *for* $p\in P$.

SKETCH OF PROOF. So as not to obscure the geometry, I will take $P=$a point and leave it out of the notation. We then consider $F\in \mathrm{Hom}_k(V, W)$ and a homotopy $f:I\to \mathrm{Hom}_k(U, W)$ with $f_0=F|U$.

Let us admit[1] that the homotopy f_t is defined on a "collar neighborhood" N (as in [7]) of U in V. By Remark 4 above, there exists a disc D^n about any point of N such that $f_t|D^n$ is a subimmersion, for $0\leqq t\leqq 1$. This suffices for the case $k=\lambda=1$ (everything is trivial when $k=0$): we construct an isotopy of $N-U$ in itself which deforms the identity to a map pulling each component of $N-U$ through such a D^n, across the foliation defined there by F (see Figure 2). Then after a preliminary homotopy defined by com-

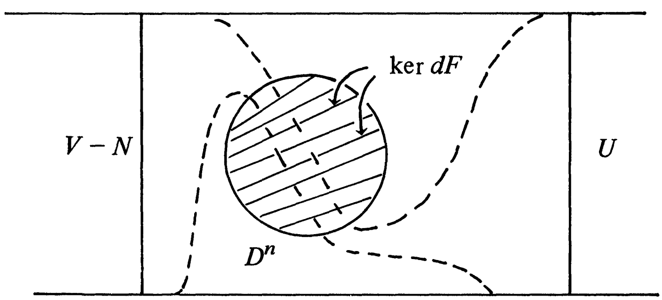

FIGURE 2

posing F with this isotopy, the stability of submersions and immersions can be used to give an initial lifting, as required. The deformation corresponding to the problem of Figure 1 might be as in Figure 3.

In general $N-U\simeq S^{\lambda-1}\times D^{n-\lambda}\times I$. The subset corresponding to the two discs would be a tubular neighborhood of the "core" $S=S^{\lambda-1}\times\{0\}\times\{\frac{1}{2}\}$. If $F|S$ is an immersion, then F subimmerses a tubular neighborhood T of S, and we proceed as before, using a preliminary isotopy which draws $N-U$ through T, across the foliation defined in T by F.

It is clearly sufficient to show that S is isotopic to a sphere immersed by F; this is proved in three steps. First, using 5.2.1 of [3] and the hypothesis $\lambda\leqq k$, the inclusion $i:S\to N-U$ is homotopic to an immersion i'

[1] This leap of faith is not required if, for $U\subset M$, $\mathrm{Hom}_k(U, W)$ is defined as inj lim $\mathrm{Hom}_k(A, W)$, where A runs through the family of open neighborhoods of U in M, and is given the quasi-topology it inherits as inj lim. See [3, §2].

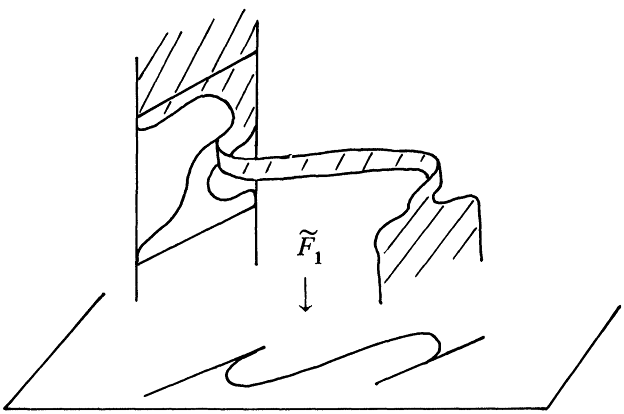

FIGURE 3

transverse to the foliation defined by F; then $F \circ i'$ is also an immersion. Next, since $\lambda \leqq a(n)$ this immersion can be C^1-approximated by an embedding i''. If the approximation is good enough, $F \circ i''$ will still be an immersion. Finally we use $\lambda \leqq a(n)$ to conclude that i and i'' are isotopic.

3. **Theorem 1′ implies Theorem 1.** Pick l sufficiently large so that geo dim $M \leqq a(n+l)$, and let $M' = M \times R^l$. This manifold satisfies the hypothesis of Theorem 1′. Give M a metric and M' the product metric. Let $p: M' \to M$ be the projection and $i: M \to M'$ the inclusion as $M \times \{0\}$.

I will prove that d induces a bijection of connected components, i.e. that $d_* : \pi_0 \operatorname{Hom}_k(M, W) \xrightarrow{\cong} \pi_0 \operatorname{Lin}_k(TM, TW)$. Higher homotopy groups can be treated analogously.

(a) d_* *is onto*. Given $H \in \operatorname{Lin}_k(TM, TW)$, the composition $H' = H \circ dp: TM' \to TW$ is homotopic to dF, for some $F \in \operatorname{Hom}_k(M', W)$, by Theorem 1′. The projection $TM \to \ker H^\perp = (\ker H')^\perp | M$ is therefore homotopic to an epimorphism $TM \to \ker dF^\perp$ covering i. It follows (see [**3**, 4.4.1], and [**6**]) that i is homotopic to a smooth map $\varphi: M \to M'$ transverse to $\ker dF$, and that H is homotopic to $d(F \circ \varphi)$, the differential of a map of rank k.

(b) d_* *is one-one*. Suppose given $f, g \in \operatorname{Hom}_k(M, W)$ and a homotopy G_t in $\operatorname{Lin}_k(TM, TW)$ joining df to dg. Composing with dp gives an arc G'_t joining $d(f \circ p)$ to $d(g \circ p)$; by Theorem 1′ the arc G'_t is homotopic with fixed endpoints to an arc dF_t, with $F_0 = f \circ p$, $F_1 = g \circ p$. It follows that the arc of projections $TM \to \ker G_t^\perp = (\ker G'_t)^\perp | M$ is homotopic to an arc of epimorphisms $TM \to \ker dF_t^\perp | M$, which we consider as an arc H_t of maps $TM \to TM'$, with H_t transverse to $\ker F_t$. *Assertion*. This arc is homotopic through such arcs to the arc of the differentials of an arc $\varphi_t: M \to M'$ with φ_t transverse to $\ker F_t$. We return to this assertion in a moment. It is easy to check, using [3] or [6] again, that i is homotopic to φ_0 through maps transverse to ker dF_0, and homotopic to φ_1 through maps transverse to ker dF_1, so that a homotopy in $\mathrm{Hom}_k(M, W)$ between f and g may be described by

$$f = F_0 \circ i \sim F_0 \circ \varphi_0 \sim F_1 \circ \varphi_1 \sim F_1 \circ i = g.$$

The assertion is an application of [3]. Let

$$A(M) = \{H \in \mathrm{Lin}(TM, TM')^I \mid H_t \text{ is transverse to } \ker dF_t \text{ for } t \in I\}$$

and $B(M)=\{f \in \mathrm{Hom}(M, M')^I | d \circ f \in A(M)\}$. Here $\mathrm{Hom}(M, M')$ is the space of smooth maps: $M \to M'$ with the C^1-compact-open topology, $\mathrm{Lin}(TM, TM')$ is the space of continuous, fiberwise linear maps: $TM \to TM'$ with the compact-open topology, and X^Y is the space of continuous maps: $Y \to X$, with the compact-open topology. It follows from [3, 2.4.1, Corollary to 3.2.3 and 3.4.1] that the "differential" $d{:}B(M) \to A(M)$ is a w.h.e.

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