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FIXED POINTS OF ENDOMORPHISMS OF COMPACT GROUPS

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1. **Introduction.** Let G be a compact, connected Lie group and denote its real Čech cohomology by $H^*(G)$. Then $H^*(G)$ is an exterior algebra with generators $1=z_0, z_1, z_2, \cdots, z_\lambda$; where, by a theorem of Hopf **[3]**, λ is equal to the rank of G (the dimension of a maximal torus). This paper announces some improvements of Hopf's result. The details will be published elsewhere.

2. **Fixed point groups.** For a set X and a function $f: X \to X$, let $\Phi(f)$ denote the set of fixed points of f: those $x \in X$ for which $f(x)=x$. If X is a topological group and f is a homomorphism, we will use the symbol $\Phi_0(f)$ for the component of the group $\Phi(f)$ which contains the identity element of X.

We consider a compact, connected Lie group G and let h be an automorphism of G. Choose algebra generators $1=z_0, z_1, z_2, \cdots, z_\lambda$ for $H^*(G)$ and let $\boldsymbol{H}^*(G)$ denote the linear span of $z_1, z_2, \cdots, z_\lambda$. The automorphism h^* of $H^*(G)$ induced by h takes $\boldsymbol{H}^*(G)$ to itself; let $\boldsymbol{h}^*$ denote the restriction of h^* to $\boldsymbol{H}^*(G)$.

Our main result is

THEOREM 1. *Let G be a compact, connected Lie group and let h be an automorphism of G. Then the rank of the Lie group $\Phi_0(h)$ is equal to the dimension of the vector space $\Phi(\boldsymbol{h}^*)$.*

Note that Theorem 1 reduces to Hopf's theorem when h is the identity function.

One might suspect that Theorem 1 is a consequence of some more intimate relationship between $H^*(\Phi_0(h))$ and $\Phi(h^*)$. However, let $g \in G$ be a regular element and define $h(x)=g^{-1}xg$, for $x \in G$, then h induces the identity isomorphism in cohomology, so $\Phi(h^*)=H^*(G)$; while $\Phi_0(h)$ is a maximal torus of G. Thus the possibilities for such a relationship are very limited.

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The proof of Theorem 1 can be reduced to two special cases: when G is abelian and when G is semisimple.

In the abelian case we can work in a more general setting. Let G be a compact, connected abelian topological group and let h be an endomorphism of G. Denote the character group of G by $G^{\wedge}$ and write the endomorphism of $G^{\wedge}$ induced by h as $h^{\wedge}$. Using the techniques of Pontryagin duality theory, we prove

THEOREM 2. *Let h be an endomorphism of a compact, connected abelian topological group G, then the dimension of the group* $\Phi_0(h)$ *is equal to the torsion-free rank of* $\Phi(h^{\wedge})$.

Let $h^{*,1}$ be the endomorphism of $H^1(G)$ induced by h. Representation theory and the continuity of Čech cohomology are used to prove

THEOREM 3. *Let h be an endomorphism of a compact, connected abelian topological group G, then the torsion-free rank of* $\Phi(h^{\wedge})$ *is equal to the dimension of the vector space* $\Phi(h^{*,1})$.

Theorems 2 and 3 imply Theorem 1 in the case that G is abelian.

If h is an automorphism of a compact, connected semisimple Lie group G, then h^m (h composed with itself m times) is an inner automorphism, for some $m \geqq 1$. We may assume that h^m is, in fact, the identity automorphism. Let Γ be the semidirect product, of G and the cyclic group of order m, induced by h. Then there is an element $\gamma \in \Gamma$ such that $h(x)=\gamma^{-1}x\gamma$ for all $x \in \Gamma_0=G$. The proof of Theorem 1, in the case that G is semisimple, now follows from Theorem 4.3 of [**1**].

3. **A bound on the rank.** Let $\mathfrak{A}$ be a simple Lie algebra and define $\rho(\mathfrak{A})$ to be the minimum rank of $\Phi(\eta)$, for all automorphisms η of $\mathfrak{A}$. The numbers $\rho(\mathfrak{A})$ are easily computed using Theorem 1 and material from [**2**].

THEOREM 4. *Let G be a compact, connected Lie group with Lie algebra* $\mathfrak{G}$. *Write*

$$\mathfrak{G} \cong \mathfrak{Z} \oplus \mathfrak{A}_1^1 \oplus \cdots \oplus \mathfrak{A}_1^{k(1)} \oplus \cdots \oplus \mathfrak{A}_u^1 \oplus \cdots \oplus \mathfrak{A}_u^{k(u)}$$

where $\mathfrak{Z}$ *is abelian,* $\mathfrak{A}_s^i \cong \mathfrak{A}_s^j \cong \mathfrak{A}_s$ *for each* $s=1,2,\cdots,u$ *and all* $i, j=1,\cdots,k(s)$, *and* $\mathfrak{A}_s^i \not\cong \mathfrak{A}_t^j$ *if* $s \neq t$. *Then*

$$\sum_{s=1}^{u} \rho(\mathfrak{A}_s) \leqq \operatorname{rank} \Phi_0(h)$$

for all automorphisms h of G.

If, for example, G is simply-connected, then there is an automorphism h of G such that rank $\Phi_0(h)$ is precisely $\sum_{s=1}^{u} \rho(\mathfrak{A}_s)$; so Theorem 4 cannot be improved in general.

COROLLARY 4.1 (DE SIEBENTHAL [**4**]). *If G is a compact, connected Lie group and there is an automorphism of G with a finite set of fixed points, then G is abelian.*

COROLLARY 4.2. *If there is an automorphism h of a compact, connected Lie group G such that $\Phi_0(h)$ is a sphere, then either G is abelian or its Lie algebra $\mathfrak{G}$ is of the form $\mathfrak{G} \cong \mathfrak{Z} \oplus \mathfrak{A} \oplus \cdots \oplus \mathfrak{A}$ where $\mathfrak{Z}$ is abelian and $\mathfrak{A}$ is a simple Lie algebra, either of type A_1 or of type A_2.*

4. **The power map.** Let G be a Lie group whose components are compact. In other words, G is an extension of a compact, connected Lie group G_0 by a discrete, but not necessarily finite, group. Define the *rank* of a component C of G to be the rank of the identity component of the centralizer of g in G, for any $g \in C$. Theorem 1 implies that the definition is independent of the choice of $g \in C$.

Define, for $k \geqq 2$, the "power map" $p_k: G \to G$ by $p_k(g) = g^k$. The component of G containing an element g is gG_0, so $p_k(gG_0) \subseteq g^k G_0$.

THEOREM 5. *Let G be a Lie group with compact components. The following are equivalent*:

(i) $p_k(gG_0) = g^k G_0$,

(ii) *the degree of the map $p_k: gG_0 \to g^k G_0$ is not zero*,

(iii) $\operatorname{rank}(gG_0) = \operatorname{rank}(g^k G_0)$.

Theorem 5 extends the main result, Theorem 5.2, of [**1**]—for compact Lie groups—to Lie groups with compact components since, when G is compact, the definition of the rank of a component given above agrees with the definition used in [**1**].

The equivalence of (ii) and (iii) follows easily from Theorem 1 above and Theorem 2.3 of [**1**]. Of course (ii) implies (i). The rest of the proof—if the degree of $p_k: gG_0 \to g^k G_0$ is zero then the dimension of $p_k(gG_0)$ is smaller than the dimension of G—reduces to the usual cases: G_0 abelian and G_0 semisimple. Following a suggestion of K. H. Hofmann, we consider the map $\varphi_k^g: G_0 \to G_0$ defined by $\varphi_k^g(x) = g^{-k}(gx)^k$ and prove the equivalent statement: if the degree of φ_k^g is zero, then $\varphi_k^g(G_0)$ has smaller dimension than G_0. In case G_0 is abelian, we again use Pontryagin duality theory. When G_0 is semisimple we can assume that g^m is in the centralizer of G_0, for some $m \geqq 1$. This permits us to construct a Lie group with identity component G_0 and only m components, apply Theorem 5.2 of [**1**] to that compact group, and then "lift" that information back to G to obtain the desired result.

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