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ORBIT STRUCTURE OF THE EXCEPTIONAL HERMITIAN SYMMETRIC SPACES. I

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In this note, we use an algebraic construction of J. Tits [7], [8] to obtain results on the orbit structure of the exceptional hermitian symmetric spaces. These results complete the explicit analysis of the orbit structure of hermitian symmetric spaces that was given by J. A. Wolf [9, pp. 321–356] for the classical cases only.

Part I is concerned with the space $E_7/E_6 \cdot SO(2)$. Part II will treat the other exceptional space, $E_6/SO(10) \cdot SO(2)$. Full details and complete proofs will appear in a longer article.

1. **J. Tits' construction of the complex Lie algebra $\mathfrak{E}_7$.** Let $\mathscr{A}$ be the algebra of 2×2 matrices with entries in $\boldsymbol{C}$ and let $\mathscr{J}$ be the 27-dimensional Jordan algebra of hermitian 3×3 matrices whose entries are complex Cayley numbers. Let $\mathscr{A}_0$ and $\mathscr{J}_0$ be the subsets of $\mathscr{A}$ and $\mathscr{J}$ consisting of matrices with zero trace. Also let $\mathrm{Der}(\mathscr{J})$ be the Lie algebra of derivations of $\mathscr{J}$. Let $\{L(A)\}(B)=A \circ B$ denote left multiplication by A in $\mathscr{J}$, and let $[a, b]=ab-ba$ for $a, b \in A$. Now define a bilinear, anticommutative multiplication $[\ ,\ ]$ on the complex vector space

$$\mathfrak{g} = (\mathscr{A}_0 \otimes \mathscr{J}) + \mathrm{Der}(\mathscr{J}) \tag{1}$$

by means of the following rules:

(a) $[D, D']$ is the usual commutator for $D, D' \in \mathrm{Der}(\mathscr{J})$.

(b) $[D, a\otimes A]=a\otimes D(A)$ for $a \in \mathscr{A}_0$, $A \in \mathscr{J}$, and $D \in \mathrm{Der}(\mathscr{J})$.

(c) $[a\otimes A, b\otimes B]=\frac{1}{2}[a, b]\otimes A \circ B+\frac{1}{2}\mathrm{Tr}(ab)[L(A), L(B)]$ for $a, b \in \mathscr{A}_0$ and $A, B \in \mathscr{J}$.

It is a theorem of J. Tits that $\mathfrak{g}$ is the complex Lie algebra $\mathfrak{E}_7$.

Let $\mathscr{A}'$ be the set of matrices in $\mathscr{A}$ with real entries and $\mathscr{A}''$ the set of matrices in $\mathscr{A}$ of the form $[\begin{smallmatrix} u & v \\ -v^* & u^* \end{smallmatrix}]$, where $u, v \in \boldsymbol{C}$ and where the asterisks indicate complex conjugation. Let $\mathscr{J}'$ be the set of matrices in $\mathscr{J}$ whose entries are real Cayley numbers. If we substitute $\mathscr{A}'$ and $\mathscr{J}'$

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for $\mathscr{A}$ and $\mathscr{J}$ in the above construction, we obtain a noncompact real form $\mathfrak{g}_0$ of $\mathfrak{g}$ with Cartan index -25. If instead we substitute $\mathscr{A}''$ and $\mathscr{J}'$, then we obtain a compact real form $\mathfrak{g}_c$ of $\mathfrak{g}$. (These results are also due to Tits [7].)

Set $\mathfrak{k}=\mathfrak{g}_0\cap\mathfrak{g}_c$ and $\mathfrak{m}_0=\mathfrak{g}_0\cap i\mathfrak{g}_c$. Then $\mathfrak{g}_0=\mathfrak{k}+\mathfrak{m}_0$ is a Cartan decomposition of $\mathfrak{g}_0$ with respect to the compact real form $\mathfrak{g}_c$ of $\mathfrak{g}$.

REMARK. Those portions of the next three sections which do not refer to § 1 are true for irreducible hermitian symmetric spaces in general. The numbered theorems concern $E_7/E_6\cdot SO(2)$ only.

2. **The exceptional space $X_c=E_7/E_6\cdot SO(2)$.** Let X_0 be the noncompact dual of the exceptional hermitian symmetric space X_c of compact type. Write X_0 as a coset space G_0/K of real Lie groups, where G_0 is the largest connected group of hermitian isometries of X_0 and where K is the isotropy subgroup of G_0 at some base point. X_c has a corresponding coset space description of the form G_c/K. G_c and K are compact; G_0 is semisimple. According to É. Cartan's classification of irreducible hermitian symmetric spaces [**2**, p. 354], the Lie algebras of G_0, G_c, and K are the algebras $\mathfrak{g}_0$, $\mathfrak{g}_c$, and $\mathfrak{k}$ constructed in §1.

3. **The almost complex structure of X_0 and X_c.** Let $\mathfrak{m}$ be the complexification of $\mathfrak{m}_0$. There is an element z in the (one-dimensional) center of $\mathfrak{k}$ such that the restriction J of ad z to $\mathfrak{m}$ satisfies $J^2=-I$. (I=identity map.) J is the almost complex structure of X_0 and X_c. Define

$$\mathfrak{m}^+ = (+i)\text{-eigenspace of } J,\quad \mathfrak{m}^- = (-i)\text{-eigenspace of } J.$$

4. **Realization of X_0 as a bounded symmetric domain.** X_c can be expressed as G/P, where G is the complexification of G_0 and where $K=G_0\cap P$. Then X_0 has a natural embedding as an open G_0-orbit on X_c. Moreover, the map $\xi:\mathfrak{m}^+\to X_c$ defined by $\xi(m)=(\exp m)P$ is a complex analytic diffeomorphism of $\mathfrak{m}^+$ onto a dense open subset of X_c containing X_0, and $\Omega=\xi^{-1}(X_0)$ is a bounded symmetric domain in $\mathfrak{m}^+$.

The domain Ω can now be described by means of a theorem of Langlands [**6**, Lemma 2]. Let σ denote conjugation of $\mathfrak{g}$ relative to $\mathfrak{g}_c$. For $u\in\mathfrak{m}^+$, define an endomorphism f_u of $\mathfrak{m}^-$ by

$$f_u(v) = [[u, \sigma u], v]\quad \text{for } v\in\mathfrak{m}^-.$$

(It is not hard to see that this is the same as the map used by Langlands.) Then the eigenvalues of f_u for each $u\in\mathfrak{m}^+$ are nonnegative real numbers.

LANGLANDS' THEOREM. $\Omega=\{u\in\mathfrak{m}^+:f_u<2I\}$.

Here "$f_u<2I$" means "all the eigenvalues of f_u-2I are negative."

If we use the construction of $\mathfrak{g}_0$ and $\mathfrak{g}_c$ in §1 to calculate the various objects defined above, we find that in the case $X_c = E_7/E_6 \cdot SO(2)$, $\mathfrak{m}^+$ and $\mathfrak{m}^-$ are isomorphic to $\mathscr{J}$. If $u \in \mathfrak{m}^+ = \mathscr{J}$ has complex conjugate u^*, then f_u can be viewed as the endomorphism of $\mathscr{J}$ defined by

$$f_u = 2\{L(u \circ u^*) - [L(u), L(u^*)]\}.$$

Hence we obtain

THEOREM 1. $\Omega = \{u \in \mathscr{J} : L(u \circ u^*) - [L(u), L(u^*)] < I\}$.

M. Koecher [**5**] and M. Ise [**3**], [**4**], working independently and using different methods, have also obtained descriptions of Ω as a subset of $\mathscr{J}$. They "look" different; however:

THEOREM 2. *The three descriptions of* Ω *are identical as point sets.*

5. **Some notational conventions.** An arbitrary matrix in $\mathscr{J}$ is of the form

$$\begin{bmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{bmatrix},$$

where the ξ_i are complex numbers and the x_i are complex Cayley numbers. The bars denote Cayley conjugation. We will use the notation

$$\xi_1 E_1 + \xi_2 E_2 + \xi_3 E_3 + F_1\{x_1\} + F_2\{x_2\} + F_3\{x_3\}$$

to represent such a matrix. If c and d are nonnegative integers such that $0 \leqq c+d \leqq 27$, let $\mathscr{J}(c, d)$ denote the set of matrices u in $\mathscr{J}$ such that f_u has c eigenvalues <2 and d eigenvalues >2 (hence $27-c-d$ eigenvalues $=2$). By Langlands' theorem, $\Omega = \mathscr{J}(27, 0)$.

Let Δ be the set of real diagonal matrices in $\mathscr{J}$. If a and b are nonnegative integers with $0 \leqq a+b \leqq 3$, let $\Delta(a, b)$ denote the $\mathrm{Ad}(K)$-orbit of the set of matrices $u = r_1E_1 + r_2E_2 + r_3E_3$ in Δ such that a of the numbers $(r_1)^2$, $(r_2)^2$, $(r_3)^2$ are <1 and b of them are >1.

6. **The G_0-orbit structure of X_c.** A close study of the eigenvalues of f_u for $u \in \mathscr{J}$, combined with some general theory in [**9**], leads to the following theorem.

THEOREM 3. *The pullbacks under* ξ *of the* G_0*-orbits on* X_c *are the sets* $\Delta(a, b)$, *where* a *and* b *are nonnegative integers such that* $0 \leqq a+b \leqq 3$. *These sets can be described in terms of the eigenvalues of* f_u, $u \in \mathscr{J}$, *as* *follows*:

$$\Delta(0,0)=\mathscr{J}(0,0),\qquad \Delta(1,0)=\mathscr{J}(17,0),\qquad \Delta(0,1)=\mathscr{J}(0,17),$$
$$\Delta(2,0)=\mathscr{J}(26,0),\qquad \Delta(0,2)=\mathscr{J}(0,26),$$
$$\Delta(1,1)=\mathscr{J}(17,9)\cup\mathscr{J}(9,17)\cup\mathscr{J}(9,9),$$
$$\Delta(3,0)=\mathscr{J}(27,0),\qquad \Delta(0,3)=\mathscr{J}(0,27),$$
$$\Delta(2,1)=\mathscr{J}(26,1)\cup\mathscr{J}(18,9)\cup\mathscr{J}(18,1)\cup\mathscr{J}(10,17)\cup\mathscr{J}(10,9)\cup\mathscr{J}(10,1),\quad \textit{and}$$
$$\Delta(1,2)=\mathscr{J}(1,26)\cup\mathscr{J}(9,18)\cup\mathscr{J}(1,18)\cup\mathscr{J}(17,10)\cup\mathscr{J}(9,10)\cup\mathscr{J}(1,10).$$

Let $S(a, b)$ *denote the* G_0*-orbit on* X_c *whose pullback under* ξ *is* $\Delta(a, b)$. *Then*

(a) *The open* G_0*-orbits on* X_c *are* $S(0, 3)$, $S(1, 2)$, $S(2, 1)$, *and* $S(3, 0)=X_0$.

(b) *The* G_0*-orbits on the topological boundary of* X_0 *in* X_c *are* $S(2, 0)$, $S(1, 0)$, *and* $S(0, 0)$. *More generally, the boundary of a typical open orbit* $S(3-b, b)$ *is the union of the orbits* $S(a', b')$ *such that* $a'+b'<3$ *and* $b'\leqq b\leqq 3-a'$.

(c) $S(0, 0)$ *is the Bergman-Šilov boundary of* X_0 *in* X_c, *the unique closed orbit.*

(d) $S(a', b')$ *is in the closure of* $S(a, b)$ *if and only if* $b'\leqq b$ *and* $a+b\leqq a'+b'$.

7. **Holomorphic arc components.** Let $\mathscr{D}=\{z\in C:|z|<1\}$. If S is a subset of X_c, then a *holomorphic arc* in S is a holomorphic map $f:\mathscr{D}\to X_c$ with image in S. A *chain of holomorphic arcs* in S is a finite sequence $\{f_1, \cdots, f_k\}$ of holomorphic arcs in S such that Image(f_j) meets Image(f_{j+1}) for $1\leqq j\leqq k-1$. Two points p, $q\in S$ are *equivalent* if there is a chain of holomorphic arcs $\{f_1, \cdots, f_k\}$ in S with $p\in$ Image(f_1) and $q\in$ Image(f_k). The equivalence classes are the *holomorphic arc components* of S in X_c. If S is open in X_c, then the holomorphic arc components of the topological boundary of S are called the *boundary components* of S.

Some computations for the complex quadric $SO(10, 2)/SO(10)\times SO(2)$, along with some results from **[9]** and the eigenvalue analysis mentioned in §6, enable us to prove Theorem 4:

THEOREM 4. *Let* a *and* b *be nonnegative integers with* $0\leqq a+b\leqq 3$. *Then the holomorphic arc components of the* G_0*-orbit* $S(a, b)$ *are symmetric spaces of rank* $a+b$ *whose pullbacks under* ξ *are the sets* $\mathrm{Ad}(k)\cdot C(a, b)$, $k\in K$, *where the subset* $C(a, b)$ *of* $\mathscr{J}$ *is described for each choice of* a *and* *b as follows*:

$$C(0,0) = \{-(E_1 + E_2 + E_3)\},$$
$$C(1,0) = \{\alpha E_1 - E_2 - E_3 : |\alpha| < 1\},$$
$$C(0,1) = \{\alpha E_1 - E_2 - E_3 : |\alpha| > 1\},$$
$$C(2,0) = \{Y = \alpha_1 E_1 + \alpha_2 E_2 - E_3 + F_3\{a_3\} : \mathrm{Tr}(Y \circ Y^*) < \min(3, 2 + |\det Y|^2)\},$$
$$C(0,2) = \{Y = \alpha_1 E_1 + \alpha_2 E_2 - E_3 + F_3\{a_3\} : 3 < \mathrm{Tr}(Y \circ Y^*) < 2 + |\det Y|^2\},$$
$$C(1,1) = \{Y = \alpha_1 E_1 + \alpha_2 E_2 - E_3 + F_3\{a_3\} : \mathrm{Tr}(Y \circ Y^*) > 2 + |\det Y|^2\},$$
$$C(a,b) = \Delta(a,b) \quad \text{when } a + b = 3.$$

In particular, the boundary components of X_0 *have pullbacks* $\mathrm{Ad}(k) \cdot C(a, 0)$, *where* $k \in K$ *and* $0 \leqq a \leqq 2$.

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