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HOMOTOPY IN HOMEOMORPHISM SPACES, *TOP* AND *PL*

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CHAPTER 0. BACKGROUND

0.1. **Introduction.** In 1926, Kneser published a paper **[39]** that received little notice until the 1950's. In this paper he proved, using conformal mapping theory, that the space $O(E^2)$ of rotations of the plane is a strong deformation retract of the space $Top(E^2)$ of all orientation preserving homeomorphisms of the plane onto itself, the topology for this space being the compact-open topology. Thus, the injection $i:O(E^2)\rightarrow Top(E^2)$ induces isomorphisms of the homotopy groups so that, as a consequence, the group of an E^2 bundle reduces to the orthogonal group.

In the 1950's, I tried to find conditions under which an open map $f:X\rightarrow I$, X compact metric, $I=[0, 1]$, each $f^{-1}(t)$ a 2-disc D^2, would be like the projection map of $D^2\times I$ onto I. It turned out (Dyer-Hamstrom **[19]**) that the fact that the space of homeomorphisms of D^2 onto itself is locally contractible enabled us to apply a selection theorem of Michael to prove that f is like a projection map if it has certain regularity properties similar to equicontinuity (see § 0.3) and that if I is replaced by a finite dimensional separable metric space, f is like the projection map of a disc bundle. Thus, in addition to their intrinsic interest, solutions to problems concerning the homotopy groups of the space of all homeomorphisms on a manifold have important consequences in the study of fibre bundles and open mappings.

It is my purpose here to survey the state of the knowledge of homotopy properties of homeomorphism spaces and, since they are clearly related, the analogous properties of certain embedding spaces. (The reader is warned that space restrictions prevent my considering codimensions other than 1.) The remainder of this chapter is devoted to definitions and Michael's theorem. In Chapter I, I consider the topological category, in Chapter II, the PL category, and in Chapter III, PL approximations and the relationship between these categories.

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0.2. **The setting; definitions and conventions.** All manifolds, unless it is specifically stated otherwise, are to be compact, with or without boundary. Let M be a manifold and let $f: M \rightarrow N$ be an embedding of M in an n-manifold N. I write $f: M \subsetneq N$ and call this embedding *proper* if $f^{-1}(\partial N) = \partial M$, where ∂ denotes boundary. *I consider such embeddings only for the case* $\dim M = n-1$ and hereafter frequently assume, without specific mention of the fact, that this is the case. The proper embedding $f: M \subsetneq N$ is *locally flat* at $f(x)$, $x \in \operatorname{int} M$ (the interior of M), if there is a homeomorphism h of E^n onto a neighborhood of $f(x)$ in N such that $h(0) = f(x)$ and $h(E^{n-1})$ is a neighborhood of $f(x)$ in $f(M)$ (where E^{n-1} denotes the hyperplane $x_n = 0$). A suitable modification of this definition is made for $x \in \partial M$. Then f is locally flat if it is locally flat at each point $f(x)$.

Let $Top(N)$ denote the identity component of the space of all homeomorphisms of a compact n-manifold N onto itself. If f is such a homeomorphism, I write $f: N \circlearrowleft$. The symbol 1_N denotes the identity homeomorphism. When no confusion exists, the subscript N will be dropped. This space is provided with the sup norm metric—i.e., if $f, g \in Top(N)$, $d(f, g) = \sup\{d(f(x), g(x)) | x \in N\}$. If M is a manifold properly and locally flatly embedded in N, let $\mathscr{E}(M, N)$ denote the space of locally flat proper embeddings of M in N that separate N, also provided with the sup norm metric. If $A \subset N$, let $Top(N; A)$ denote the subspace of $Top(N)$ consisting of those homeomorphisms that leave A pointwise fixed. In what follows, A will usually be either ∂N or a properly and locally flatly embedded manifold. Similarly, if $A \subset M$, a properly and locally flatly embedded manifold, let $\mathscr{E}(M, N; A)$ denote the subspace $\mathscr{E}(M, N)$ consisting of those homeomorphisms that leave A pointwise fixed. If it makes sense, let $O(N)$ denote the space of orientation preserving rigid motions of N. It is important to note that *all of these spaces are topologically complete.* Since $Top(N)$ is a group (under the operation of composition) that acts freely and transitively on N, its homotopy properties are of special interest in fibre bundle theory.

If X is a metric space, let $B(x, \varepsilon)$ denote the open ball with center x and radius ε and, if $A \subset X$, $N(A, \varepsilon)$ the ε-neighborhood of A. The space X is *k-connected* for a nonnegative integer k if $\pi_j(X) = 0$ for $j \leqq k$ and *locally k-connected* (LC^k) if for each $x \in X$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that every map $f: S^j \rightarrow B(x, \delta)$ ($j \leqq k$) is homotopic to 0 in $B(x, \varepsilon)$. The space X is *locally contractible* if for each $x \in X$ and $\varepsilon > 0$, there is a $\delta > 0$ such that $B(x, \delta)$ is contractible in $B(x, \varepsilon)$.

It is important to note here that since $Top(N)$ is a group whose operation is composition of homeomorphisms, if it is LC^k or locally contractible at 1_N, it is so at every element. Also, the easily proved fact that $d(f, g)=d(fh, gh)$ implies that for $Top(N)$, the δ in the definition of LC^k and local contractibility depends only on ε—i.e., $Top(N)$ is LC^k if and only if for each ε there is a δ such that if $f \in Top(N)$, every map of S^j $(j \leqq k)$ into $B(f, \delta)$ is homotopic to 0 in $B(f, \varepsilon)$ and if we exhibit such a δ for 1_N it works for every f.

In E^n, D^n denotes the closed unit ball centered at the origin, 0, and S^{n-1} denotes its boundary. The ball D^{n-1} is identified with $D^n \cap E^{n-1}$. As usual, I denotes the interval $[0, 1]$.

Two homeomorphisms $f, g: M \hookrightarrow N$ are *isotopic* if there is a homeomorphism $F: M \times I \hookrightarrow N \times I$ such that $F(x, 0)=(f(x), 0)$, $F(x, 1)=(g(x), 1)$ and F commutes with projection on the second factor—i.e., $F(x, t) \in N \times t$. (Such an F is called *level preserving with respect to I.*) If F moves no point by as much as ε, f and g are *ε-isotopic*. The homeomorphism F defines a path $\{F_t | t \in I\}$ in $\mathscr{E}(M, N)$—$F_t(x)=\pi F(x, t)$, where π is the projection on the first factor. I sometimes use the notation $\{F_t\}$ to name the isotopy. If F extends to $G: N \times I \circlearrowleft$, then f and g are called ambient isotopic.

If g is fixed and F depends continuously on f, F is called *canonical*. Thus, when we say, for f in $\mathscr{E}(M, N)$, that f is canonically isotopic to 1_M, we are, in effect, saying that $\mathscr{E}(M, N)$ is contractible. If we say that for each $\varepsilon>0$, there is a $\delta>0$ such that each f in $N(1_M, \delta)$ is canonically isotopic to 1_M in $N(1_M, \varepsilon)$, we are saying that $\mathscr{E}(M, N)$ is locally contractible at 1_M.

If for each $f \in \mathscr{E}(M, N)$ and $\varepsilon>0$, there exists a $\delta>0$ such that if $g \in \mathscr{E}(M, N)$ and $d(f, g)<\delta$, then there is a $G \in Top(N)$ such that $d(G, 1_N)<\varepsilon$ and $Gf|M=g$, then $\mathscr{E}(M, N)$ has the *complete regularity property*. If, for each f, G depends continuously on g, then $\mathscr{E}(M, N)$ has the *local canonical extension property*. More precisely, if Z is a metric space and for each $f \in \mathscr{E}(M, N)$ and $\varepsilon>0$, there is a $\delta>0$ such that if $\varphi: Z \rightarrow N(f, \delta)$ maps Z continuously into the δ-neighborhood of f in $\mathscr{E}(M, N)$, then there is a map $\Phi: Z \rightarrow N(1_N, \varepsilon)$ such that $\Phi(z)f|M=\varphi(z)$, then $\mathscr{E}(M, N)$ has the *local canonical extension property with respect to Z*. If for each $\varphi: Z \rightarrow \mathscr{E}(M, N)$ there is a map $\Phi: Z \rightarrow Top(N)$ such that $\Phi(z)$ extends $\varphi(z)$, then $\mathscr{E}(M, N)$ has the *canonical extension property with respect to Z*. If $\Phi(z)$ is defined only over a component Q of $N-M$, $\mathscr{E}(M, N)$ has the canonical extension property over Q with respect to Z.

Let $\mathscr{E}_0(M, N)$ denote the subspace of $\mathscr{E}(M, N)$ consisting of those embeddings that are restrictions to M of elements of $Top(N)$. If $i: Top(N; M) \hookrightarrow Top(N)$ is the injection and $r: Top(N) \rightarrow \mathscr{E}_0(M, N)$ is induced by restriction—i.e., $r(h)=h|M$, then $Top(N; M) \hookrightarrow Top(N) \rightarrow \mathscr{E}_0(M, N)$ is a (Serre) fibration if $\mathscr{E}_0(M, N)$ has the canonical extension property with respect to D^k for each k. This is the usual covering homotopy property. Note that if $r(h)=f$, $r^{-1}(f)$ is homeomorphic to $Top(N; M)$. If this is a fibration, then the homotopy exact sequence can be used to obtain information about the homotopy properties of these spaces.

0.3. **A selection theorem of Michael.** The following is a weakened version of Michael's theorem that is useful for my purposes.

THEOREM [**44**]. *If X is a topologically complete metric space, Y is metrizable and has covering dimension at most $n+1$, and $f:X\to Y$ is an open, homotopy n-regular (see below) surjection, then for each closed $Z\subset Y$ and map $g:Z\to X$ such that $fg=1_Z$, there is an open U in Y such that $Z\subset U$ and an extension $G: U\to X$ of g such that $fG=1_U$. If the point inverses under f have vanishing homotopy groups in dimensions at most n, then U may be taken to be Y. (The maps g and G are cross sections.)*

The map $f: X\to Y$ is *homotopy n-regular* if it is open and, for each $y\in Y$, $x\in f^{-1}(y)$ and $\varepsilon>0$, there is a $\delta>0$ such that any map of S^k, $k\leqq n$, into $B(x, \delta)\cap f^{-1}(y')$, $y'\in Y$, is homotopic to 0 in $B(x, \varepsilon)\cap f^{-1}(y')$. Note that this implies that each $f^{-1}(y)$ is LC^n.

I now describe the setting in which this theorem is used. There is a map $f: X\to Y$, X and Y as described above, and a compact space K such that for each y, there is a homeomorphism f_y of K onto $f^{-1}(y)$. The map f is required to be *completely regular*—i.e., for each y and $\varepsilon>0$, there is a $\delta>0$ such that if $y'\in Y$ and $d(y', y)<\delta$, then there is an ε-homeomorphism of $f^{-1}(y)$ onto $f^{-1}(y')$. From the map f there is obtained a map $f^*: X^*\to Y$ where, for each y, $f^{*-1}(y)$ is the space (homeomorphic to $Top(K)$) of all homeomorphisms of K onto $f^{-1}(y)$. Suppose it is known that $Top(K)$ is LC^n. The complete regularity makes it easy to prove that for (X^*, f^*, Y), the hypotheses of Michael's theorem are satisfied so that for each y, there is a neighborhood U of y and a map $G_U: U\to X^*$ such that $f^*G_U=1_U$. Thus there is a homeomorphism $g_U: U\times K\to f^{-1}(U)$ defined by $g_U(z, x)=G_U(z)(x)$. Note that if $Top(K)$ is n-connected, then U may be taken to be all of Y. Note further that if $L\subset K$ and $g: Y\times L\to X$ is a homeomorphism such that $\pi g^{-1}|g(Y\times L)=f|g(Y\times L)$, then if $Top(K; L)$ is LC^n and for each y, the homeomorphism in the definition of complete regularity can be chosen to take $g(y, x)$ onto $g(y', x)$ for each x in L, then U and G_U can be chosen so that g_U extends $g|U\times L$.

Complete regularity turns up naturally. If K is a manifold of dimension $n\leqq 3$, then (modulo the Poincaré conjecture) a homotopy n-regular map is completely regular ([**21**] and [**22**]). In this case, the homotopy 0-regular maps are the regular maps studied by White, Whitney, Whyburn and others ([**58**], [**59**], and [**60**]).

We see, then, that the homotopy properties of $Top(N)$ can give information about when maps whose point inverses are all copies of N are like projection maps of products and it will be seen that this, in turn, can be used to get information about canonical extension properties. (See §1.3, [22] and [27] for further details.)

Chapter I. The category *TOP*

1.1. Some elementary examples; the Alexander trick.

1.1.1. *The unit interval.*

THEOREM 1.1.1. *$Top(I)$ is homeomorphic to $Top(I; \partial I)$ and is contractible and locally contractible.*

PROOF. Let h be an orientation preserving homeomorphism of I onto itself and, for each $x \in I$ and $t \in I$, let $h_t(x)=th(x)+(1-t)x$. It is clear that $\{h_t\}$ is an isotopy from h to 1_I and that it is canonical. Also, $d(h, 1)<\delta$ implies that $d(h_t, 1)<\delta$, so that $Top(I)$ is contractible and locally contractible.

1.1.2. *The unit circle.* Coordinatize S^1 by the reals mod 1. Obviously, $Top(S^1; p)$ is homeomorphic to $Top(I)$ for any point p of S^1. Therefore, if $h \in Top(S^1)$, there is an isotopy in $Top(S^1; h(1))$ from h to a rotation. This isotopy is clearly canonical, so that we have

THEOREM 1.1.2. *The space $O(S^1)$ is a strong deformation retract of $Top(S^1)$. Since $O(S^1)$ is homeomorphic to S^1, it follows that $\pi_1(Top(S^1))=Z$ and $\pi_i(Top(S^1))=0$ for $i>1$.*

1.1.3. *The unit ball; the Alexander trick.* Consider first $Top(D^n; S^{n-1}\cup 0)$. If h is in this space, let h^* denote h extended by the identity to all of E^n. For each $t \in I$, and $x \in D^n$, let $h_t(x)=th^*(x/t)$. Then $\{h_t\}$ is a canonical isotopy in $Top(D^n; S^{n-1}\cup 0)$ from h to 1. If $d(f, g)<\varepsilon$, then $d(f_t, g_t)<\varepsilon$. Therefore

THEOREM 1.1.3.1. *$Top(D^n; S^{n-1}\cup 0)$ is contractible and locally contractible (Alexander [3]).*

This technique is the by now celebrated Alexander trick and is a very useful tool.

Observe that if $h \in Top(D^n; S^{n-1})$ and moves the origin, h can be canonically isotoped in $Top(D^n; S^{n-1})$ to an element of $Top(D^n; S^{n-1}\cup 0)$ so that

THEOREM 1.1.3.2. *$Top(D^n; S^{n-1})$ is contractible and locally contractible*

A *radial homeomorphism* of D^n is a homeomorphism that takes each radius linearly onto a radius. Denote by $Top_R(D^n)$ the subspace of $Top(D^n)$ consisting of all radial homeomorphisms. The argument above proves

THEOREM 1.1.3.3. *$Top_R(D^n)$ is a strong deformation retract of $Top(D^n)$. Since $Top_R(D^n)$ is homeomorphic to $Top(S^{n-1})$, it follows that if $O(S^{n-1})$ is a strong deformation retract of $Top(S^{n-1})$, then $O(D^n)$ is a strong deformation retract of $Top(D^n)$.*

Also, since each element of $Top(S^{n-1})$ extends uniquely via a radial homeomorphism to an element of $Top(D^n)$, it is not hard to prove

THEOREM 1.1.3.4. *$Top(D^n;\ S^{n-1}) \subsetneq Top(D^n) \rightarrow Top(S^{n-1})$ is a fibration* [**40**].

1.2. **The unit 2-sphere—an example of a technique.** Since what goes on here is fairly easily visualized and the technique can be used in a much more general setting, I shall pay particular attention to this example.

It will be clear that whatever is done to $Top(S^2;p)$ will be canonical (p an arbitrary point of S^2). By canonically moving h to a homeomorphism leaving the antipode, q, of p fixed, we can restrict our attention to $Top(S^2;p\cup q)$.

Let S^1 be the great circle half way between p and q and D the disc in S^2 bounded by S^1 and containing p. For each $h\in Top(S^2;p\cup q)$, let $r_h=\sup\{r \mid \exists$ circle with center p and radius r in $h(D)\cap D\}$. The number r_h depends continuously on h [**38**] and the circle S_h with center p and radius $r_h/2$ lies in the interior of $h(D)\cap D$. If we canonically isotope h to a homeomorphism for which $r_h/2$ is, say, 1, we may assume for homotopy purposes that the circle S_0^1 centered at p of radius 1 bounds a disc D_0 lying in the interior of D and of each $h(D)$. We have seen that the set of h for which this is true is a strong deformation retract of $Top(S^2;p\cup q)$.

Let A denote the annulus bounded by $S_0^1\cup S^1$ and A_h that bounded by S_0^1 and $h(S^1)$. Several possibilities may occur.

Case 1. $\mathscr{E}(S^1, S^2-D_0)$ has the canonical extension property over A.

Case 2. $\mathscr{E}(S^1, S^2-D_0)$ has the canonical extension property of A with respect to S^k for each k.

Case 3. $\mathscr{E}(S^1, S^2-D_0)$ has the local canonical extension property over A.

Case 4. $\mathscr{E}(S^1, S^2-D_0)$ has the local canonical extension property over A with respect to S^k for each k.

If any of these cases occurs, then there may be applied what I call the Roberts trick, since it was communicated to me by J. H. Roberts and is what he had in mind in [**50**]. Let F_h denote the canonical extension of

$h|S^1$ (guaranteed by one of the four cases listed above) to a homeomorphism of A onto A_h. Then F_h extends through h (to S^2-D) and the radial extension (to D_0) to a homeomorphism $\tilde{F}_h: S^2 \circlearrowleft$. The homeomorphism $\tilde{F}_h^{-1}h$ is 1 on S^1 so, by the Alexander trick, it is canonically isotopic to 1, leaving the points of S^1 fixed. Similarly, since $\tilde{F}_h$ takes S_0^1 onto itself, it is canonically isotopic to a radial extension of $\tilde{F}_h|S_0^1$. Thus $h=\tilde{F}_h(\tilde{F}_k^{-1}h)$ is canonically isotopic to the radial extension of $\tilde{F}_h|S_0^1$ which, in turn (see §1.1.3), is canonically isotopic to a rotation.

Depending on which of the four cases (if any) is true, one of the following conclusions holds. I point out that *all* of the cases are true, so that the following are, in fact, theorems.

Case 1.

THEOREM 1.2.1. *$O(S^1)$ is a strong deformation retract of $Top(S^2; p\cup q)$.*

Since what we have done is canonical with respect to p, we have

THEOREM 1.2.2. *$O(S^2)$ is a strong deformation retract of $Top(S^2)$.*

THEOREM 1.2.3. *$O(E^2)$ is a strong deformation retract of $Top(E^2)$ (compact-open topology).*

This last result is of particular interest since Browder [**6**] has shown that for some n, $O(E^n)$ and $Top(E^n)$ are not homotopically equivalent. Thus E^n bundles do not always contain D^n bundles. (The n is not exhibited.)

Case 2.

THEOREM 1.2.4. *$Top(S^2; p\cup q)$ is weakly homotopically equivalent to $O(S^1)$—i.e., the injection from the space of rotations leaving $p\cup q$ fixed into $Top(S^2; p\cup q)$ induces isomorphisms of homotopy groups.*

THEOREM 1.2.5. *$O(S^2)$ and $Top(S^2)$ are weakly homotopically equivalent. Roughly, if $f: S^k\to Top(S^2)$, then $f(x)$ is canonically isotopic to a rotation and if $f(x)$ is already a rotation, the isotopy leaves it fixed.*

Since $O(S^2)$ is homeomorphic to P^3, real projective 3-space, S^3 is a covering space of P^3, and we have the Hopf fibration $S^1 \subsetneq S^3 \to S^2$, the homotopy groups of $Top(S^2)$ are known modulo those of S^2 and S^3.

THEOREM 1.2.6. *$\pi_1(Top(S^2))=\pi_1(P^3)=Z_2$, $\pi_2(Top(S^2))=0$ and*

$$\pi_k(Top(S^2)) = \pi_k(S^3) = \pi_k(S^2)$$

for $k>2$.

Case 3. In the performance of the Roberts trick, if both h and F_h are near the identity, then all the maps in the isotopy from h to a rotation are near the identity.

THEOREM 1.2.7. *$Top(S^2)$ is locally contractible.*

Case 4. The comments for Case 3 apply here also.

THEOREM 1.2.8. *$Top(S^2)$ is LC^k for all k.*

This example illustrates a close relationship between the homotopy properties and the canonical extension properties. It is from this point of view that I have studied these problems. It was essentially from this point of view that Kneser (as I mentioned in §0.1) proved these theorems for S^2. Although his proof is not exactly like that outlined above, the essential ideas are the same. He used the *continuity* property of conformal mappings on the annulus (see **[41]** or **[57]**) to obtain the necessary canonical extension properties of Cases 1 and 2 above.

1.3. *$Top(A)$* **and canonical extensions over** *A*. Without conformal mapping theorems, which are certainly not topological, we can still obtain

THEOREM 1.3.1. *$\mathscr{E}(S^1, S^2-D_0)$ has the canonical extension property over A with respect to S^k, so that Case* 3 *of* §1.2 *holds.*

INDICATION OF PROOF. Consider $f:S^k\to\mathscr{E}(S^1, S^2-D_0)$. Let A_x be the annulus bounded by S_0^1 and $f(x)(S^1)$. There is a level preserving (with respect to S^k) embedding $f^*:\partial A\times S^k \hookrightarrow S^2\times S^k$ defined by $f^*(y, x)=(f(x)(y), x)$ if $y\in S^1$ and $f^*(y, x)=(y, x)$ if $y \in S_0^1$. What is needed is a level preserving embedding $F^*:A\times S^k\hookrightarrow S^2\times S^k$ that extends $f^*|S^1\times S^k$ and takes $S_0^1\times S^k$ onto itself. Then for each x, F_x^* is the canonical extension we want.

The embedding F^* is obtained by means of Michael's selection theorem (§0.3). Look at $X=\bigcup (A_x\times\{x\})\subset S^2\times S^k$ and $p:X\to S^k$, the projection onto the second factor. That p is completely regular, so that for y near x, there is a small homeomorphism from A_x to A_y extending $f(y)f(x)^{-1}$ follows from basic theorems in plane topology. It will be seen that $Top(A; S^1)$ is LC^k for each k and homotopically trivial. It has already been observed that it is topologically complete. Michael's theorem applied to the map $p^*:\_\to S^k$ such that for each x, $p^{*-1}(x)$ is the space of all homeomorphisms of A onto $A_x\times\{x\}$ taking $y\in S^1$ onto $(f(x)(y), x)$, then yields, as explained in §0.3, the required homeomorphism F^*. Thus, the problem of computing the global homotopy groups of $Top(S^2)$ is reduced to that for the global and local homotopy groups of $Top(A)$.

THEOREM 1.3.2. *$\mathscr{E}(S^1, S^2-D_0)$ has the local canonical extension property over A with respect to S^k, so that Case* 4 *of* §1.2 *holds.*

PROOF. Things are done a little differently here but still depend strongly on the fact that $Top(A; S^1\cup S_0^1)$ is LC^k for each k. Recall that, as explained in §0.2, this means that for each ε there is a δ such that if f is any map of S^k into $B(1, \delta)$, f is homotopic to 0 in $B(1, \varepsilon)$. What is needed (the notation as above) is an embedding $F^*: A \times S^k \to S^2 \times S^k$ that extends $i \cup f^* | (S_0^1 \cup S^1) \times S^k$ (i is the injection) and moves no point as much as ε if f^* moves no point as much as δ. What follows outlines an argument whose details (for dimension 3) are given in [22]. For $k=1$, we obtain F^* by first applying the local version of Michael's theorem to the map $p^*: \_ \to S^1$ for which $p^{*-1}(x)$ is the space of homeomorphisms of A onto $A_x \times \{x\}$ taking y to (y, x) if $y \in S_0^1$ and to $(f(x)(y), x)$ if $y \in S^1$. From Michael's theorem, we get a sequence $a_0 < a_1 < \cdots < a_n = a_0$ of points of S^1 and for each i, a level preserving homeomorphism $F_i: A \times [a_i, a_{i+1}] \to p^{-1}[a_i, a_{i+1}]$ extending $i \cup f^*$ and moving no point very far. The product structure and the local connectivity of $Top(A; S^1 \cup S_0^1)$ is used to fit these homeomorphisms together to get F^*. Let $F_{i,t}: A \subsetneq S^2$ be defined by $F_{i,t}(y) = \pi F_i(y, (1-t)a_i + ta_{i+1})$, where π is projection on the first factor and $\bar{t} = (1-t)a_i + ta_{i+1}$. Then $F_{i,a_i}^{-1} F_{i-1,a_i}: A \circlearrowleft$ is an element of $Top(A; S^1 \cup S_0^1)$ near 1 so is isotopic to 1 via an isotopy $\{H_t\}$ every element of which is near 1 ($H_0 = F_{i,a_i}^{-1} F_{i-1,a_i}$ and $H_1 = 1_A$). Define $F_i^*: A \times [a_i, a_{i+1}] \to S^2 \times [a_i, a_{i+1}]$ by $F_{i,\bar{t}}^*(y) = F_{i,\bar{t}} H_t(y)$, where the $\bar{t}$ stands for $(1-t)a_i + ta_{i+1}$. It is easy to check that these homeomorphisms agree on the a_i and do not move any point very far. I let the reader fill in the ε's and δ's.

If $k > 1$, note first that if $Top(N)$ is LC^{k-1}, then the space of level preserving (with respect to S^{k-1}) homeomorphisms of $N \times S^{k-1}$ is LC^0 (see [22]). Use induction, assuming that the local extensions exist with respect to S^{k-1}. Consider S^k as $S^{k-1} \times I$ with $S^{k-1} \times 0$ and $S^{k-1} \times 1$ reduced to points. Let $p: X \to I$ take $A_x \times \{x\}$ onto the I-coordinate of x. Except for $t=0$ or $t=1$, $p^{-1}(t)$ is homeomorphic to $A \times S^{k-1}$. Michael's theorem may now be applied to $p^*: \_ \to I$, where $p^{*-1}(t)$ is the space of level preserving homeomorphisms of $A \times S^{k-1}$ onto $p^{-1}(t)$, with suitable restrictions to $\partial A \times S^{k-1}$. For some sequence $0 = a_0 < \cdots < a_n = 1$, we get $F_i: A \times S^{k-1} \times [a_i, a_{i+1}] \subsetneq p^{-1}[a_i, a_{i+1}]$. We can fit these together as indicated above to obtain F^*.

What we need, then, is information about the homotopy properties of $Top(A)$.

THEOREM 1.3.3. *$Top(A; \partial A)$ is LC^k for each k.*

PROOF. An argument similar to the one above can be used. Consider A as $S^1 \times I$ and for some two points a and b of S^1, let $I_a = \{a\} \times I$ and $I_b = \{b\} \times I$. Suppose that $f: S^k \to Top(A; \partial A)$ is such that no $f(x)(I_b)$ meets I_a (I_a and I_b will play the roles of S^1 and S_0^1 above). Let D_0 and D_1 be the discs into which $I_a \cup I_b$ decomposes A. Since, by the Alexander trick, $Top(D_i; \partial D_i)$ is LC^k for each k, an argument similar to that outlined above for $\mathscr{E}(S^1, S^2 - D_0)$ gives the local extension property for

$\mathscr{E}(I_b, A-I_a; \partial A)$ with respect to S^k. (Conformal mapping theory would permit us to drop the "with respect to S^k".) Thus, for $x \in S^k$, there is a canonical homeomorphism $F_x \in Top(A; \partial A)$ such that $F_x|I_a = 1_{I_a}$ and $F_x|I_b = f(x)|I_b$. The Roberts trick gives the required homotopy from f to 0. (Note that $Top(D^2; \partial D^2)$ is homeomorphic to $Top(A; \partial A \cup I_a)$.)

THEOREM 1.3.4. *$Top(A; \partial A)$ is homotopically trivial.*

The proof of this theorem requires more work, but the proof uses Michael's theorem and is essentially elementary. The details are much like those used in the computation of $\pi_i(Top(T^2))$ [24].

1.4. **How to apply these techniques to a general situation.** The arguments used in §§1.2 and 1.3 have general application. Suppose that the properly, locally flatly embedded $(n-1)$-manifold M separates N, that $\mathscr{E}(M, N)$ has the complete regularity property and that $Top(N; M)$ is LC^k for each k. The argument used for $\mathscr{E}(S^1, S^2-D_0)$ proves that $\mathscr{E}(M, N)$ has the local canonical extension property with respect to S^k for each k. The same argument works to prove that $\mathscr{E}(M, N)$ has the canonical extension property with respect to D^k for each k. Prove it first for $k=1$, then, as above, use induction by considering D^k as $D^{k-1}\times I$. Here we do not have to worry about ε's and δ's. Thus, as explained in §0.2, $Top(N; M) \subsetneq Top(N) \to \mathscr{E}_0(M, N)$ is a fibration. If $Top(N; M)$ is homotopically trivial, then $\mathscr{E}(M, N)$ has the canonical extension property with respect to S^k.

The reader who examines the induction argument will see first that if it is known that $\mathscr{E}(M, N)$ has the local canonical extension property with respect to S^k, then if the Roberts trick is applicable, it can be proved that $Top(N)$ is LC^k and second, that *if* it is known that $Top(N; M)$ is LC^k, then the induction argument proves that $\mathscr{E}(M, N)$ has the local canonical extension property with respect to S^{k+1}. For the case $Top(S^2)$, the applicability of the Roberts trick depended on $Top(A)$ and this, in turn, depended on $Top(D^2)$. So here, the applicability of the Roberts trick depends on $Top(M\times I; \partial(M\times I))$ and this, in turn, will depend on Top (some other, hopefully less complicated manifold) et cetera.

1.5. $Top(M^2)$.

THEOREM 1.5.1. *If M is a compact 2-manifold, $Top(M)$ is LC^k for each k.*

PROOF. Use the standard handlebody decomposition. This consists of disjoint discs $B_1, \cdots, B_n$, disjoint discs $C_1, \cdots, C_m$ such that for each i, j, $B_i \cap C_j$ is an arc or empty and C_j meets exactly two B_i's and disjoint discs $E_1, \cdots, E_p$ such that each E_k meets each B_i and C_j in an arc or not at all and meets exactly three B_i's and three C_j's. These discs are obtained from a triangulation T of M by taking the stars of the barycenters of the simplices of T in the second barycentric subdivision of T. Thus, the B's are the stars of the vertices, the C's the stars of the barycenters of the edges, the E's the stars of the barycenters of the faces.

Consider B_1 and a disc B_1' such that $B_1 \subset \operatorname{int} B_1'$ and $B_1' \cap B_i = \varnothing$, $i \neq 1$. For $f: S^k \to Top(M)$, the image sufficiently near 1, $h = f(x)$, we have $h^{-1}(B_1) \subset \operatorname{int} B_1'$. There is a canonical (with respect to S^k) homeomorphism $F_h \in Top(M)$, also near 1, such that $F_h | B_1 = h^{-1} | B_1$ and $F_h | M - B_1' = 1$. Use the Alexander trick to get a canonical small isotopy from F_h to 1. Then hF_h, which is 1 on B_1, canonically isotopes to h via a small isotopy. Do this for all B_i so that h canonically isotopes, via a small isotopy, to h_1, which leaves each B_i pointwise fixed. The argument may be repeated for, say, C_1. Suppose that $C_1 \cap B_i$ and $C_1 \cap B_j$ are arcs. Choose a disc C_1' containing C_1 such that $C_1' \cap C_k = \varnothing$, $k \neq 1$, $C_1' \cap B_i$ and $C_1' \cap B_j$ are arcs with $C_1 \cap B_i$ and $C_1 \cap B_j$ in their interiors, C_1' meets no other B's and $C_1 \cap C_1' = (C_1 \cap B_i) \cup (C_1 \cap B_j)$. Then h_1 can be canonically isotoped, via a small isotopy, to h_2, which is 1 on each B and each C. Finally, the Alexander trick may be applied to each E_k so that h is canonically (with respect to S^k) isotopic, via a small isotopy, to 1. The reader can easily supply the ε's and δ's. This argument is like Kister's argument in [37] and indicates that for 2-manifolds, we need only prove the local canonical extension properties and the LC^k property for certain simple situations and that a general argument can then be used to prove that $Top(M)$ is LC^k.

The global results require more work but the homotopy groups of $Top(M^2)$ have all been computed with the help of Michael's theorem.

THEOREM 1.5.2. *If M^2 has nonempty boundary or has genus greater than 1 if orientable or greater than 2 if nonorientable, then $Top(M^2; \partial M^2)$ is homotopically trivial. For projective space, P^2, $\pi_i(Top(P^2)) = \pi_i(P^2)$ for $i > 2$, $\pi_2(Top(P^2)) = 0$ and $\pi_1(Top(P^2)) = Z_2$. For the Klein bottle, K, $\pi_i(Top(K)) = 0$, $i > 1$, and $\pi_1(Top(K)) = Z$. Also, $\pi_i(Top(T^2)) = \pi_i(T^2)$ for each i. (See* **[24]**, **[25]**, *and* **[26]**.)

For $x \in M^2$, it is also useful to know about $Top(M^2; x)$. Quintas and McCarty (**[43]** and **[48]**) discuss this. See Quintas **[49]** for a general discussion of $Top(M^2)$.

1.6. **3-manifolds.** Suppose that $\dim M = 2$ and $\dim N = 3$. That $\mathscr{E}(M, N)$ has the complete regularity property was proved in **[22]**. The proof depends, of course, on work of Bing and Moise (**[4]**, **[5]**, and **[45]**).

THEOREM 1.6.1. *If N is a compact 3-manifold, $Top(N)$ is LC^k for each k.*

We can proceed exactly as in §1.5, paying attention to the comments in §1.4. The standard handlebody decomposition, obtained in the same way as that for M^2 from a triangulation T of N, consists of four families

$\{B_{ij}:j=1, \cdots, n_i\}$, $i=0, 1, 2, 3$, of disjoint (tame) balls, each B_{ij} the star in the second barycentric subdivision of the barycenter of an i-simplex of T. Each B_{1j} meets exactly two B_{0j}'s and these in discs; each B_{2j} meets three B_{0j}'s and three B_{1j}'s, each intersection a disc, the intersection with the union an annulus; each B_{3j} meets four B_{0j}'s, six B_{1j}'s and four B_{2j}'s, each intersection a disc, the intersection with the union a 2-sphere.

Suppose that $f: S^k \to Top(N)$ has image near 1. Consider first the B_{0j}'s. By taking a suitable ball B'_{0j}, $B_{0j} \subset \text{int } B'_{0j}$, we can proceed exactly as we did for M^2 to canonically (with respect to S^k) isotope a homeomorphism $h = f(x)$ to a homeomorphism h_1 that moves no point in any B_{0j}, the isotopy being small. As explained in §1.4, we can do this if

(0) $\quad Top(S^2 \times I; \partial(S^2 \times I))$ is LC^k for each k.

Similarly, by taking a suitable ball B'_{1j}, $B_{1j} \subset B'_{1j}$, we can canonically isotope h_1 via a small isotopy to a homeomorphism h_2 leaving each B_{0j} and each B_{1j} pointwise fixed if

(1) $\quad Top(S^1 \times D^2; \partial(S^1 \times D^2))$ is LC^k for each k.

(The set $B'_{1j} - \text{int } B_{1j}$ will be homeomorphic to $S^1 \times D^2$.)

Then we can canonically isotope h_2 via a small isotopy to a homeomorphism h_3 leaving each B_{ij}, $i \leqq 2$, pointwise fixed since (for suitable B'_{2j}) $B'_{2j} - \text{int } B_{2j}$ is homeomorphic to a union of disjoint balls. Finally, h_3 is isotopic to 1 via the Alexander trick applied to the B_{3j}. Thus $Top(N)$ is LC^k for all k.

The case (1) is proved in much the same way that it was proved in §1.3 that $Top(A; \partial A)$ is LC^k. For some $a, b \in S^1$, let I_a and I_b denote $\{a\} \times D^2$ and $\{b\} \times D^2$, then repeat the argument in §1.3.

To prove the case (0), consider two discs $D_0 \subset \text{int } D_1$ in S^2. Then $D_0 \times I$ and $(S^2 - \text{int } D_0) \times I$ are balls whose union is $S^2 \times I$. The set $K = (D_1 \times I) - \text{int}(D_0 \times I)$ is homeomorphic to $S^1 \times D^2$, so $Top(K; \partial K)$ is LC^k for each k. Thus, we have, for $\mathscr{E}(\partial D_1 \times I, (S^2 - D_0) \times I)$, the local canonical extension property with respect to S^k. The Roberts trick may now be applied to prove that $Top(S^2 \times I; \partial(S^2 \times I))$ is LC^k for all k.

The arguments outlined in §1.4 now apply to prove that $Top(M^3; M^2) \subsetneq Top(M^3) \to \mathscr{E}_0(M^2, M^3)$ is a fibration and that for each k, $\mathscr{E}(S^2, S^3)$ has the extension property with respect to S^k. The proof of the local connectivity is like Kister's in [**37**] but the general arguments can also be found in [**22**]. See also [**18**].

Global results are harder to come by. Scott and Akiba ([**55**] and [**2**]), have announced results from which it follows that $O(S^3)$ and $Top(S^3)$ are weakly homotopically equivalent and that $\pi_i(Top(D^2 \times S^1; \partial(D^2 \times S^1))) = 0$ for each i, but proofs of these facts have not yet appeared.

1.7. **Some important recent results.** The Edwards-Kirby theorem. Friberg **[16]** has recently given an extremely elegant, purely topological proof that $O(S^2)$ is a strong deformation retract of $Top(S^2)$. Earlier, Morton proved **[47]**, using conformal mapping theorems, that if M is a disc with holes, then $Top(M; \partial M)$ is contractible. It would appear that this theorem for arbitrary 2-manifolds and other results may now be amenable to Friberg's technique.

In **[9]**, Černavskiĭ proved

THEOREM 1.7.1. *For all compact manifolds N, $Top(N)$ is locally contractible.*

The proof, while extremely complicated, is essentially elementary and involves the construction, for $h \in Top(N)$, of a canonical sequence of expanding and shrinking homeomorphisms going from h to 1. All these homeomorphisms may be PL. Černavskiĭ's work is discussed in some detail by Rushing in **[52]**.

In **[7]**, Morton Brown proved

THEOREM 1.7.2 (*The Schoenflies theorem*). *If $h: S^n \hookrightarrow S^{n+1}$ is locally flat, then h extends to $H: S^{n+1} \circlearrowleft$. The local flatness condition implies the existence of a bicollar. If this collar is canonical, then so is H. This was proved in* **[17]** *by Gould, who observed that Brown's proof is canonical.* (*See also Huebsch-Morse* **[33]**.)

More precisely,

THEOREM 1.7.3. *If $h: S^n \times [-1, 1] \hookrightarrow S^{n+1}$ is an embedding, then $h|S^n \times 0$ extends canonically with respect to h to a self homeomorphism of S^{n+1}.*

Note that this does not say that h itself extends nor does it say that $\mathscr{E}(S^n, S^{n+1})$ has any of the canonical extension properties I have been discussing. The canonical collar is necessary to the argument.

In **[14]**, Edwards and Kirby also prove that $Top(N)$ is locally contractible for all compact manifolds. Neither proof of this theorem requires a triangulation for N. The Edwards-Kirby proof is much in the spirit of what I have been discussing, so I shall give it further attention. (See Rushing **[52]**.)

EDWARDS-KIRBY MAIN LEMMA. *Consider the space*

$$\mathscr{E}^*(D^k \times 4D^n, D^k \times E^n; \partial D^k \times 4D^n).$$

(The asterisk indicates that there is no local flatness condition assumed.) If $\varepsilon > 0$, there is a $\delta > 0$ such that if $h \in \mathscr{E}^*$ and moves no point as much as δ,

then there is a canonical isotopy $\{h_t\}$ *such that* $h_0=h$, $h_t|\partial(D^k\times 4D^n)=h|\partial(D^k\times 4D^n)$, $h_1|D^k\times D^n=1$ *and* h_t *moves no point as much as* ε. *The lemma easily reduces to a similar lemma for the subspace* $\mathscr{E}^{**}$ *of* $\mathscr{E}^*$ *consisting of those embeddings that leave* $C=[\frac{1}{2}, 1]D^k\times 3D^n$ *pointwise fixed.*

The main tool in the proof is the construction of a local canonical extension $\Phi(h): D^k\times E^n\to D^k\times E^n$ such that

$$\Phi(h) \mid (\partial D^k \times E^n) \cup (D^k \times (E^n - 3\operatorname{int} B^n)) = 1$$

and $\Phi(h)|D^k\times D^n=h|D^k\times D^n$. The map Φ (of a neighborhood of 1 into a neighborhood of 1) can be constructed via Kirby's main diagram ([**36**], [**52**]) in two different ways. One version uses as its main tools an immersion of T^n-D^n into 3 int D^n (T^n is the n-torus) and the canonical Schoenflies theorem. The other version omits these devices and uses instead an expanding and shrinking process à la Černavskiĭ. Once $\Phi(h)$ is constructed, a version of the Roberts trick proves the lemma.

In a triangulated manifold N—say, for convenience, without boundary—the handles constructed in the standard handlebody decomposition (as described in §§1.5 and 1.6 for M^2 and M^3) are of a form that is amenable to the construction of $\Phi(h)$. Suppose that T is a triangulation of N and σ is the barycenter of a k-simplex of T. One of the k-handles is H, the star of σ in the second barycentric subdivision of T. The handle H is homeomorphic to $D^k\times D^{n-k}$ in such a way that it intersects the union of the j-handles, $j<k$, in a copy of $\partial D^k\times D^{n-k}$. Suppose that $h\in Top(N)$, is near 1_N and has been canonically isotoped via a small isotopy to h_k, which is 1 on all j-handles, $j<k$. Let H' be a copy of $D^k\times 5D^{n-k}$ containing H as $D^k\times D^{n-k}$ and such that H' meets no other k-handles and meets the union of the j-handles, $j<k$, in a copy of $\partial D^k\times 5D^{n-k}$. The construction in the main lemma now yields a canonical $\Phi(h)$, $\Phi(h)=1$ on $N-H'$ and $\Phi(h)=h_k^{-1}$ on H. Exactly as described in §§1.5 and 1.6 for M^2 and M^3, we get an isotopy from h_k to a homeomorphism leaving H and all j-handles, $j<k$, pointwise fixed. A repetition of this yields the required canonical isotopy from h to 1.

It must be pointed out that the proof for nontriangulated manifolds is considerably more complicated, but it is certainly in the spirit of what I have outlined above. Also, Edwards and Kirby have a more general result, which they use the main lemma to prove.

THEOREM 1.7.4. *If* C *is compact and* $C\subset\operatorname{int} U\subset U\subset N$, *then for any homeomorphism* $h: U\to N$ *sufficiently near* 1, *there is a canonical small isotopy* $\{h_t\}$ *such that* $h_0=h$, $h_1|C=1$ *and for some compact neighborhood* V *of* C, *each* $h_t|N-V=1$.

Seebeck and Wright have each used this in modified forms to obtain useful results. Seebeck in [**56**] states his modified form as follows.

THEOREM 1.7.5. *If M is a closed PL manifold and $\varepsilon>0$, there is a $\delta>0$ such that if $h: M\times[-\frac{1}{2}, \frac{1}{2}] \subsetneq M\times[-1, 1]$ is within δ of 1, then there is an ε-isotopy $\{g_t\}$ such that $g_0=h$, $g_1|M\times 0=h|M\times 0$ and for each t, $g_t|M\times\{-\frac{1}{2}, \frac{1}{2}\}=h|\{-\frac{1}{2}, \frac{1}{2}\}$.*

He then proves

THEOREM 1.7.6. *For each ε there is a δ such that if $h: M\times 0 \subsetneq M\times[-1, 1]$ is locally flat and within δ of 1 and $h(M\times 0)\cap(M\times 0)=\varnothing$, then there is a homeomorphism $H: M\times I\to M\times I$ such that for each x, $H(x, 0)=x$, $H(x, 1)=h(x)$ and* diam $H(x\times I)<\varepsilon$.

He uses this to prove

THEOREM 1.7.7. *If M (of codimension 1) is a closed PL manifold in the interior of a PL manifold N, then M has a collar on one side if it can be approximated by locally flat embeddings on that side.*

Wright proves in **[61]**

THEOREM 1.7.8. *If $n\neq 4$,* dim $M=n-1$ *and $\varepsilon>0$, then there is a $\delta>0$ such that if $h: M\times 0\subsetneq M\times E^1$ is a local flat embedding within δ of 1, then h extends to $H: M\times E^1\to M\times E^1$ such that H is within ε of 1.*

He then goes on to prove a complete regularity property.

THEOREM 1.7.9. *If $n\neq 4$, $M=\partial Q^n$ is locally flatly embedded in N and admits a PL structure, then $\mathscr{E}(M, N)$ has the complete regularity property. (Note that $\mathscr{E}(S^{n-1}, S^n)$ satisfies this property.)*

Thus, using the fact that *Top*$(N; M)$ is LC^k for each k, we get, following §1.4,

THEOREM 1.7.10. *Under the conditions of Theorem 1.7.9, Top$(N; M)\subsetneq$ Top$(N)\to\mathscr{E}_0(M, N)$ is a fibration and $\mathscr{E}(M, N)$ has the local canonical extension property with respect to S^k.*

There also follows readily

THEOREM 1.7.11. *$\mathscr{E}(M, N)$ is LC^k for each k.*

PROOF. If $f: S^k\to\mathscr{E}(M, N)$ takes S^k into a small neighborhood of 1, then the local canonical extension property yields $F: S^k\to$ *Top*(N) taking S^k into a small neighborhood of 1. Since *Top*(N) is LC^k, there is a small homotopy from F to 0. Restriction gives the homotopy of f.

Černavskiĭ has also proved many similar and related results. Announcements may be found in **[10]**.

CHAPTER II. THE CATEGORY *PL*

2.1. **Definitions and background.** In this chapter, all manifolds are PL with a fixed PL structure and the homeomorphisms are those elements of $Top(N)$ or $\mathscr{E}(M, N)$ that are also PL. Instead of Top and $\mathscr{E}$, I use the symbols PL and $\mathscr{E}PL$. The symbol Δ^k denotes the standard k-simplex and I^n denotes $[-1, 1]^n$ with a standard PL structure. I use S^n to denote ∂I^{n+1} or $\partial\Delta^{n+1}$, whichever is more convenient. The elements of $PL(M, N)$ are required to be locally *unknotted*—i.e., for $x \in \text{int } M$ and $h \in \mathscr{E}PL(M, N)$, there are PL balls $A \subset h(M)$ and $B \subset N$ such that $x \in \text{int } A$, $B \cap h(M) = A$, and (B, A) is PL homeomorphic to the standard ball pair $(\Sigma\Delta^{n-1}, \Delta^{n-1})$, where Σ denotes suspension. Suitable modifications apply to $x \in \partial M$. There are now several texts in PL topology. See, for instance, Hudson [**32**].

Homotopy can be considered in two different ways. We can consider the *subspaces* $Top_{PL}(N)$ and $\mathscr{E}_{PL}(M, N)$ of $Top(N)$ and $\mathscr{E}(M, N)$ consisting of the PL homeomorphism. Černavskiĭ's proof (see [**9**]) shows that $Top_{PL}(N)$ is locally contractible.

THEOREM 2.1.1. *For $h \in Top_{PL}(N)$ and near* 1, *there is a canonical small isotopy $H: N \times I \circlearrowleft$ such that each H_t is PL, $H_0 = h$ and $H_1 = 1$. The homeomorphism H need not be PL.*

In the PL category, the usual setting and the one I shall use is, for $PL(N)$, the semisimplicial complex whose k-simplices are level preserving PL homeomorphisms $f: \Delta^k \times N \circlearrowleft$. The ith boundary operator ∂_i is defined by $\partial_i f = f | (i\text{th face of } \Delta^k) \times N$. The semisimplicial complexes $PL(N; M)$ and $\mathscr{E}PL(M, N)$ are similarly defined. A semisimplicial map from one complex to another takes k-simplices to k-simplices and commutes with each ∂_i. I will not be concerned with technical details here but refer the reader to [**42**] and [**51**].

It is possible in this setting to get a suitable homotopy theory. For the present purposes it suffices to note a few pertinent facts. These semisimplicial complexes have the homotopy type of CW-complexes. Thus weak homotopy equivalence is homotopy equivalence and $PL(N)$ may be called contractible if all its homotopy groups vanish. An element of $\pi_k(PL(N))$ is represented by a PL level preserving homeomorphism $f: \Delta^k \times N \circlearrowleft$ such that $f | \partial\Delta^k \times N = 1$. The homeomorphisms f and g are homotopic if there is a PL level preserving (with respect to Δ^k and I) homeomorphism $H: \Delta^k \times N \times I \circlearrowleft$ such that $H_0 = f$, $H_1 = g$, and $H | \partial\Delta^k \times N \times I = 1$. The complex $PL(N)$ is LC^k if for each ε, there is a δ such that if $f: \Delta^j \times N \circlearrowleft$, $j \leqq k$, represents $\pi_j(PL(N))$ and moves no point as much as δ, then there is a PL homeomorphism $F: \Delta^j \times N \times I \circlearrowleft$ such that $F_1 = f$, $F_0 = 1$, F moves no point as much as ε and $F | \partial\Delta^j \times N \times I = 1$.

REMARK. In order to compare the categories Top and PL, it will be convenient to also consider $Top(N)$ as a semisimplicial complex. Here the k-simplices are level preserving homeomorphisms $f: \Delta^k \times N \circlearrowleft$.

2.2. The Alexander trick, Hudson-Zeeman theorems and $PL(S^1)$.

THEOREM 2.2.1. *$PL(I^n; \partial I^n)$ is contractible and locally contractible.*

PROOF. Let $f: \Delta^k \times I^n$ represent $\pi_k(PL(I^n; \partial I^n))$. Let σ be the barycenter of Δ^k and in $\Delta^k \times I^n \times I$, let p denote $(\sigma, 0, 0)$. Let T be a triangulation of $\Delta^k \times I^n$ with respect to which f is simplicial and T_1 a triangulation of $\Delta^k \times I^n \times I$ that has $p * T$ (the cone of T) as a subdivision. Define $F: \Delta^k \times I^n \times I \to \Delta^k \times I^n \times I$ as follows. If $(x, y, 1) \in \Delta^k \times I^n \times I$,

$$F(t(x, y, 1) + (1-t)p) = t(x, f_x(y), 1) + (1-t)p,$$

roughly, if $a = (x, y, 1)$, F takes the interval from p to a linearly onto the interval from p to $f(a)$. Since $f|\partial(\Delta^k \times I^n) = 1$, we can extend F by the identity to all of $\Delta^k \times I^n \times I$ to get the required PL homeomorphism. Thus f is homotopic to 0. Clearly, if f moves no point as much as ε, neither does any F_t.

THEOREM 2.2.2. *$PL(S^1; p)$ is contractible and locally contractible.*

PROOF. It is clear that $PL(S^1; p)$ is isomorphic to $PL(I^1; \partial I^1)$.

Something is needed here to take the place of Michael's theorem. It comes from the covering isotopy theorems of Hudson and Zeeman ([**30**] and [**31**]).

THEOREM 2.2.3. *If $f: \Delta^k \times M \hookrightarrow \Delta^k \times N$ is a proper, locally unknotted, level preserving PL embedding, then f extends to a PL level preserving $F: \Delta^k \times N \circlearrowleft$.*

Here, the local unknottedness condition is special and implies that there is a PL level preserving collar $f^*: \Delta^k \times M \times [-1, 1] \to \Delta^k \times N$, where $f^*(x, y, 0) = f(x, y)$. For proper f, unknottedness at each level implies this if dim $N \leqq 3$ and codimension of $M = 1$. If the codimension of $M \geqq 3$, this follows from Zeeman's unknotting theorem.

As an elementary application, I indicate how to prove the following

THEOREM 2.2.4. *$PL(S^1)$ is homotopically equivalent to S^1.*

PROOF. We must show that a natural PL map of S^1 into $PL(S^1)$ is a homotopy equivalence. Consider a PL analog of $O(S^1)$ obtained as follows, with $S^1 = \partial \Delta^2$ and Δ^1 identified with I. Parametrize S^1 in a PL way by the reals mod 1—say by means of a PL map from I to S^1 identifying 0 and 1.

Let $\varphi:\partial\Delta^2\times I\circlearrowleft$ be a PL level preserving homeomorphism such that $\varphi_0=\varphi_1$ and $\varphi_t(0)=t$. (The φ_t are "rotations".) Let $OPL(S^1)$ be the semi-simplicial complex whose k-simplices are PL homeomorphisms $f:\Delta^k\times\partial\Delta^2\circlearrowleft$, each f_t being some φ_s. It is not hard to prove that in the semi-simplicial sense, $\pi_i(OPL(S^1))=0$ if $i>1$ and $\pi_1(OPL(S^1))=Z$. It remains to prove that the injection $i:OPL(S^1)\subsetneq PL(S^1)$ is a homotopy equivalence.

Let $f:\Delta^k\times\partial\Delta^2\circlearrowleft$ represent an element of $\pi_k(PL(S^1))$. There is a level preserving homeomorphism $f^*:\Delta^k\times 0\times I\subsetneq\Delta^k\times\partial\Delta^2\times I$ such that $f_0^*(x,0)=f(x,0)$, $f_1^*(x,0)=(x,0)$ if $k>1$, and $f_1^*(x,0)=(x,nx)$ for some integer n if $k=1$. (Recall that $\Delta^1=I$ and $\partial\Delta^2$ is parametrized by the reals mod 1.) Then the Hudson-Zeeman theorems apply ($M=0$, $N=\partial\Delta^2$) to yield an extension of f^* to $F^*:\Delta^k\times\partial\Delta^2\times I\circlearrowleft$ such that $F_0^*=f$ and $F^*|\partial\Delta^k\times\partial\Delta^2\times I=1$. If $k>1$, $F^*|\partial\Delta^k\times\partial\Delta^2\times 1$ represents an element of $\pi_k(\partial\Delta^2;0)$ so is homotopic to 0. Thus the PL product structure can be used, much as explained in §1.3 to adjust F^* so as to be 1 on $\Delta^k\times\partial\Delta^2\times 1$. If $k=1$, observe that there are two PL homeomorphisms $g_1, g_2:\Delta^1\times\partial\Delta^2\times 1\circlearrowleft$, g_1 taking $(s,x,1)$ onto $(s,\varphi_{ns}(x),1)$, g_2 taking $(s,x,1)$ to $F^*(s,x,1)$. These agree on $\Delta^1\times 0\times 1$. Thus $g_2^{-1}g_1$ represents an element of $\pi_1(\partial\Delta^2;0)$, so is homotopic to the constant function $1:\Delta^1\times\partial\Delta^2\times 1\circlearrowleft$. If we compose each element of the homotopy with g_2 we get that g_1 is homotopic to g_2. Therefore the product structure may be used to adjust F^* so that it is g_1 on $\Delta^1\times\partial\Delta^2\times 1$. Thus f is homotopic to a representative of $\pi_k(OPL(S^1))$. A similar argument will prove that if α, β, represent elements of $\pi_1(OPL(S^1))$ and α is homotopic to β in $PL(S^1)$, then α is homotopic to β in $OPL(S^1)$. Thus $OPL(S^1)$ and $PL(S^1)$ are homotopically equivalent.

2.3. **Theorems in dimensions 2 and 3.** The PL analog of the complete regularity property for $PL(M,N)$, $\dim N=3$ is essentially contained in [**22**]. Craggs ([**12**] and [**13**]) also proves this as well as the fact that $PL(N)$ is LC^0. The arguments used in the category *TOP* can be repeated in *PL* to prove

THEOREM 2.3.1. *If* $\dim N\leq 3$, *then* $PL(N)$ *is* LC^k [**34**].

When Michael's theorem is used in *TOP*, the Hudson-Zeeman theorems are used in *PL*, the fitting together through the product structure being done exactly as outlined in §1.3.

The proofs in [**26**] for the global results if dim $N=2$ can also be linearized. In particular

THEOREM 2.3.2. $O(S^2)$ *is homotopically equivalent to* $PL(S^2)$. (*See Scott* [**54**], *Akiba* [**1**] *and Morlet* [**46**].)

The announcements of Scott and Akiba mentioned earlier also state a similar result for S^3.

2.4. **Higher dimensional, codimension 1 results.** Completely general results for higher dimensions cannot be obtained. First, Browder's theorem for *TOP* mentioned earlier holds also for *PL*—i.e., for some n, $O(S^n)$ and $PL(S^n)$ are not homotopically equivalent. Second, the local unknottedness condition makes the Hudson-Zeeman theorems difficult to apply. Third, the existence of exotic triangulations of T^n, $n \geqq 5$, shows that $PL(T^n)$ is not LC^0 (Theorem C of Kirby's notes **[36]**).

In the positive sense, Siebenmann (as communicated to me by Edwards) has recently proved for $n \geqq 5$ and N without boundary that $PL(N)$ is LC^k if $H^{2-j}(N; Z_2)=0$ for each $j \leqq k$. This is closely related to the handlebody structure of N (see Kirby's notes). The condition is also necessary, so in particular, if N is without boundary, $PL(N)$ is not LC^2. Also, for any codimension, $PL(N; M)$ is LC^k for each k if and only if $H^k(N, M; Z_2)=0$ for each $k \leqq 2$. This is more often true.

Chapter III. Approximations

3.1. **Approximations in** $Top(N^3)$. In **[5]** and **[45]**, Moise and Bing prove that if $f \in Top(N^3)$ and $\varepsilon > 0$, there is an $f^* \in PL(N^3)$ such that $d(f, f^*) < \varepsilon$. There is a uniform version of this. In this and the following section, dim $N=3$, dim $M=2$, and N has a fixed PL structure in which M is a polyhedron.

Theorem 3.1.1. *If $f: \Delta^k \times N \circlearrowleft$ represents an element of $\pi_k(Top(N))$, then for each $\varepsilon > 0$, there is a PL $f^*: \Delta^k \times N \circlearrowleft$ such that $d(f, f^*) < \varepsilon$.*

Proof. If $k=1$, the Moise-Bing results yield, for each $x \in \Delta^1$, a PL approximation φ_x to f_x. The product structure may be used to obtain points $0 = x_0 < \cdots < x_n = 1$ and PL homeomorphisms $\Phi_i: [x_i, x_{i+1}] \times N \circlearrowleft$ such that for each i, Φ_i approximates $f | [x_i, x_{i+1}] \times N$, $\Phi_{ix} = \varphi_{x_i}$ for $x_i \leqq x \leqq x_{i+1}$ and $\Phi_{10} = 1_N$. The fact that $PL(N)$ is LC^0 may now be used, as explained in §1.3, to fit the Φ_i together at the x_i to obtain the required homeomorphism f^*. For $k > 1$, use induction, giving Δ^k a product structure $\Delta^{k-1} \times I$. Use the induction hypothesis to get $0 = x_0 < \cdots < x_n = 1$ and PL $\Phi_i: \Delta^{k-1} \times [x_i, x_{i+1}] \times N \circlearrowleft$ approximating $f | \Delta^{k-1} \times [x_i, x_{i+1}] \times N$, $\Phi_i | \partial \Delta^{k-1} \times [x_i, x_{i+1}] \times N = 1$, $\Phi_{ix} = \varphi_{x_i}$ and $\Phi_{10} = 1$. These may be fitted together as before.

A modification of this argument proves

Theorem 3.1.2. *If* dim $N=3$, *$Top(N)$ is homotopically equivalent to $PL(N)$—i.e., in the semisimplicial sense, $i: PL(N) \subsetneq Top(N)$ induces isomorphisms of homotopy groups.*

PROOF. Since $Top(N)$ is LC^k for each k, it follows readily from the above that f is homotopic to f^* in $Top(N)$ so that $i_*: \pi_k(PL(N)) \to \pi_k(Top(N))$ is a surjection. The approximation method can be used to prove that if $f^*: \Delta^k \times N \circlearrowleft$ represents an element of $\pi_k(PL(N))$ and is homotopic to 0 in $Top(N)$, then f^* is homotopic to 0 in PL—i.e., we construct a PL approximation to $F: \Delta^k \times N \times I \circlearrowleft$. Thus $i_*: \pi_k(PL(N)) \to \pi_k(Top(N))$ is an isomorphism.

3.2. **Approximations in $\mathscr{E}(M^2, N^3)$.** Bing proved in **[4]** that every element of $\mathscr{E}(M, N)$ can be approximated by an element of $\mathscr{E}PL(M, N)$. Suppose that $F \in Top(N)$ and $f = F|M$. Sanderson proved in **[53]** that if $f^* \in \mathscr{E}PL(M, N)$ is a sufficiently close approximation to f, then f^* extends to $F^* \in PL(N)$, a close approximation to F. See also **[12]** and **[13]**. There is a uniform version of Sanderson's theorem.

THEOREM 3.2.1. *If $F: \Delta^k \times N \circlearrowleft$ represents an element of $\pi_k(Top(N))$ and $f = F|\Delta^k \times M$, then for each $\varepsilon > 0$, there is a $\delta > 0$ such that if h represents an element of $\pi_k(\mathscr{E}PL(M, N))$ and $d(f, h) < \delta$, then there is a representative $H: \Delta^k \times N \circlearrowleft$ of $PL(N)$ that extends h and is such that $d(F, H) < \varepsilon$.*

PROOF. For $k=1$, the idea of the proof is simple. Apply Sanderson's theorem to get a suitable extension over each $x \times N$ then use the Hudson-Zeeman theorems to get $0 = x_0 < \cdots < x_n = 1$ and $H_i: [x_i, x_{i+1}] \times N \circlearrowleft$ extending $h|[x_i, x_{i+1}] \times M$, the intervals $[x_i, x_{i+1}]$ so short that H_i moves no point very much. Since $PL(N; M)$ is LC^0, these homeomorphisms can be fitted together as in §1.3 and above to obtain the required H. An induction argument now works as before.

There is also a uniform version of Bing's theorem.

THEOREM 3.2.2. *If $f: \Delta^k \times M \to \Delta^k \times N$ represents an element of $\pi_k(\mathscr{E}(M, N))$ and $\varepsilon > 0$, then there is a representative h of an element of $\pi_k(\mathscr{E}PL(M, N))$ such that $d(f, g) < \varepsilon$.*

PROOF. It is convenient to think of Δ^k as the union of two copies of Δ^k, $\Delta^k_1 \cup \Delta^k_2$, $\Delta^k_1 \cap \Delta^k_2 = \Delta^{k-1}$. The results in Chapter I give $F_i: \Delta^k_i \times N \circlearrowleft$ extending $f|\Delta^k_i \times M$, $F_i = 1$ on $\partial\Delta^k_i \times N$. Then there are PL homeomorphisms $H_i: \Delta^k_i \times N \circlearrowleft$ that approximate F_i and restrict to 1 on $\partial\Delta^k \times N$. Usually, these cannot be made to agree on $\Delta^{k-1} \times N$. However, the $H_i|\Delta^{k-1} \times M$ do approximate $F_1|\Delta^{k-1} \times M$, so by the uniform version of Sanderson's theorem, $H_i|\Delta^{k-1} \times M$ extends to a PL $G_i: \Delta^{k-1} \times N \circlearrowleft$ and G_1 is close to G_2. Since $PL(N)$ is LC^{k-1}, there is a PL $\Phi: \Delta^{k-1} \times N \times I \circlearrowleft$, $\Phi_0 = G_1$, $\Phi_1 = G_2$, and $\Phi|\partial\Delta^{k-1} \times N \times I = 1$. The product structure may now be used to fit H_1 and H_2 together via Φ.

All results in this section can be proved for dim $N=2$.

3.3. **Higher dimensional approximation theorems.** Again, for higher dimensions of N but codimension 1 for M, results are difficult to obtain. As a positive result, Connell [**11**] showed that stable and hence all homeomorphisms (Kirby [**35**]) of S^n can be approximated by piecewise linear ones. Cantrell and Rushing use Connell's result in [**8**] to prove that if $f: S^{n-1} \subsetneq S^n$, $n \geqq 5$, is locally flat, then there is a small ambient isotopy taking f to a locally unknotted embedding.

A negative result is an example described by Edwards in [**15**]. There is a manifold N with two PL structures, and a codimension 1 submanifold M such that $\mathscr{E}PL(M, N)$ is not LC^0. Also, if M has two PL structures, M_θ and M_Σ, the topological embedding $1: M_\theta \times 0 \subsetneq M_\Sigma \times E^1$ cannot be approximated by a PL embedding.

CONCLUSION. I have completely omitted any discussion of codimensions other than 1. For dimension 3, most of the theorems here hold if codimension is 2 or 3, except that an unknottedness condition is frequently needed. For higher dimensions, nothing much is known about codimension 2. In codimension 3, everything seems to be true. Edwards, Miller, Bryant-Seebeck, Černavskiĭ, Rushing, Homma, to name only a few, have made valuable contributions.

Also, there is the general problem of determining the topological properties of these spaces. For instance, $Top(M^2)$ is an ANR (Luke-Mason), $Top(M) \times l_2$ is homeomorphic to $Top(M)$ (Geohegan) and $Top_{PL}(M)$ is an ANR. The reader is strongly urged to consult the papers of the authors named.

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