BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 1, January 1974

ON HILBERT TRANSFORMS ALONG CURVES

BY ALEXANDER NAGEL,[1] NESTOR RIVIÈRE[2] AND STEPHEN WAINGER[1]

Communicated by E. Stein, June 1, 1973

Let $\gamma(t)$, $-\infty<t<\infty$, be a smooth curve in R^n. For f in $C_0^\infty(R^n)$ set

$$(1)\qquad Tf(x) = \lim_{\varepsilon\to\infty, N\to\infty} \int_{\varepsilon\leqq|t|\leqq N} \frac{f(x-\gamma(t))}{t}\,dt.$$

Tf is the Hilbert transform of f along the curve $\gamma(t)$. E. M. Stein **[2]** raised the following general question: For what values of p and what curves $\gamma(t)$ is Tf a bounded operator in L^p? If $\gamma(t)$ is a straight line it is well known that T is bounded for $1<p<\infty$. Stein and Wainger **[3]** proved that the operator is bounded for $p=2$ if

$$\gamma(t) = (|t|^{\alpha_1}\operatorname{sgn} t, \cdots, |t|^{\alpha_n}\operatorname{sgn} t), \qquad \alpha_i > 0.$$

Here we show that Tf is a bounded operator in L^p for some p other than 2 and some nontrivial, nonlinear γ's. We prove

THEOREM 1. *Let* $\gamma(t)=(|t|^{\alpha_1}\operatorname{sgn} t, |t|^{\alpha_2}\operatorname{sgn} t)\alpha_1>0$, $\alpha_2>0$. *Then* Tf *is bounded in* L^p *for* $\frac{4}{3}<p<4$.

SKETCH OF THE PROOF. The transformation (1) may be expressed as a multiplier transformation. In our case,

$$(2)\qquad (Tf)\hat{}(x, y) = m(x, y)\hat{f}(x, y)$$

where

$$(3)\quad m(x, y) = \lim_{\varepsilon\to\infty, N\to\infty} \int_{\varepsilon\leqq|t|\leqq N} \exp\{i\,|t|^{\alpha_1}\operatorname{sgn} tx + i\,|t|^{\alpha_2}\operatorname{sgn} ty\}\,\frac{dt}{t}$$

($\hat{}$ denotes Fourier transform).

By a change of variables we may assume $\alpha_1=1$ and $\alpha_2\geqq1$. Furthermore we may assume $\alpha_2>1$, for otherwise we have the case that $\gamma(t)$ is a straight

---

AMS (*MOS*) *subject classifications* (1970). Primary 42A18, 42A40.

[1] Partially supported by an NSF grant at the University of Wisconsin.

[2] Partially supported by an NSF grant at the University of Minnesota, GP-15832.

line. Thus in (3) we take $\alpha_1=1$ and $\alpha_2=\alpha>1$. Clearly m is odd and $m(rx, r^\alpha y)=m(x, y)$, $r>0$. By using the method of steepest descents and integration by parts we obtain

THEOREM 2. *$m(x, y)$ is infinitely differentiable away from the line $y=0$.*

For $0\leqq|y|/x^\alpha\leqq 1$,

$$m(x, y) = m_1(x, y) + m_2(x, y) + m_3(x, y),$$

where, if

$$\lambda = |y|\, x^{-\alpha} \quad \text{and} \quad \beta = (a-1)^{-1}$$

$$m_1(x, y) = \begin{cases} \sum_{j=1}^{n} A_j \lambda^{\beta/2+\eta_j} \exp(i\lambda^{-\beta}\nu_j), & y \geqq 0, \\ 0, & y \leqq 0, \end{cases}$$

$$m_2(x, y) = \begin{cases} \sum_{j=1}^{m} B_j \lambda^{\beta/2+\rho_j} \exp(i\lambda^{-\beta}\xi_j), & y \leqq 0, \\ 0, & y \geqq 0, \end{cases}$$

$m_3(x, y)$ has continuous second order partial derivatives away from the origin. Here A_j and B_j are complex numbers $\eta_j\geqq 0$, $\rho_j\geqq 0$, and ν_j and ξ_j are real.

We shall consider a multiplier of the form $n(x, y)=g(y/x^\alpha)$ where

$$g(\lambda) = \begin{cases} \lambda^{\beta/2} \exp(i\lambda^{-\beta})\omega(\lambda), & \text{if } \lambda > 0, \\ 0, & \text{if } \lambda \leqq 0, \end{cases}$$

where ω is C^∞, has support in $[-1, 1]$ and is identically 1 near $\lambda=0$. Theorem 2 implies that $m(x, y)$ is a finite sum of multipliers each of which may be treated in the same way as $n(x, y)$. Set

$$g_z(\lambda) = \begin{cases} \lambda^{z\beta} \exp(i\lambda^{-\beta})\omega(\lambda), & \lambda \geqq 0, \\ 0, & \lambda \leqq 0, \end{cases}$$

and $n_z(x, y)=g_z(y/x^\alpha)$.

We wish to show

$$n_{1/2} \text{ is a bounded multiplier on } L^p \text{ for } \tfrac{4}{3} < p < 4.$$

Clearly $n_{0+it}(x, y)$ is a bounded multiplier on L^2 (with norm uniformly bounded in t). Hence, in view of the interpolation theorem for analytic families of operators, to prove $n_{1/2}$ is a bounded multiplier on L^p, $\frac{4}{3}<p<4$, it suffices to prove

THEOREM 3. *$n_{\sigma+it}$ is a bounded multiplier on L^p, $1<p<\infty$ for $\sigma>1$, with a bound that is independent of t.*

Theorem 3 will in turn follow by arguments similar to Rivière [1], if one can prove the following

LEMMA. *Let* $\psi(r)$ *be in* $C^\infty[0, \infty)$ *with support in* $[\frac{1}{2}, 2]$, $\rho(x,y)=(x^{2\alpha}+y^2)^{1/2\alpha}$, *and* $\phi(x,y)=\psi(\rho(x,y))$. *For* δ *positive and small set* $l=\frac{1}{2}(1+1/\alpha)+\alpha$ and $k=(\alpha+1)/2$.

Then

$$\text{(i)} \qquad \int_{R^2}(|x|^{2k} + |y|^{2l})\,|(n_{\sigma+it}\phi)^\vee(x, y)|^2\, dx\, dy \leqq C$$

and

$$\text{(ii)} \qquad \int_{R^2}(|x|^{2k} + |y|^{2l})\,|(n_{\sigma+it}\phi h_{s,u})^\vee(x, y)|^2\, dx\, dy \leqq C[\rho(s, u)]^2.$$

$h_{s,u}(x, y)=e^{i(xs+yu)}-1$. ($^\vee$ denotes inverse Fourier transform).

Lemma 2 is proved by (a) proving appropriate analogues of (i) and (ii) if $k=m+it$, m a nonnegative integer, $l=1+it$, and $l=it$, and then (b) using the Phragmén-Lindelöf theorems. Details will appear elsewhere.

REFERENCES

1. N. M. Rivière, *Singular integrals and multiplier operators*, Ark. Mat. **9** (1971), 243–278.

2. E. M. Stein, *Some problems in harmonic analysis*, Proc. Internat. Congress Math. (Nice, 1970), vol. 1, Gauthier-Villars, Paris, 1971, pp. 173–190.

3. E. M. Stein and S. Wainger, *The estimation of an integral arising in multiplier transformations*, Studia Math. **35** (1970), 101–104. MR **42** #904.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINNESOTA 55455