BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 1, January 1974

COHOMOLOGY AND WEIGHT SYSTEMS FOR NILPOTENT LIE ALGEBRAS

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Communicated by Dock Rim, July 17, 1973

1. This paper announces results concerning the cohomology groups $H^*(N, N)^T$ where N is in a certain class of finite-dimensional nilpotent Lie algebras over a field k and T is an abelian Lie algebra faithfully represented as a maximal diagonalizable algebra of derivations of N; we shall refer to such an N as a T-algebra. The additional hypotheses to be placed on the pair N, T are inspired by the case when T is a Cartan subalgebra and $T+N=B$ is a Borel subalgebra of a complex semisimple Lie algebra. In that case Kostant has shown [**2**] that $H^i(N, N)^T=0$ for $i\geqq 2$ and the authors applied this result in [**3**] to conclude that $H^*(B, B)=0$. (A similar argument shows $H^*(P, P)=0$ for P parabolic.) Here we are concerned with the relations between the vanishing of $H^i(N, N)^T$, especially for $i=2$, and the structure of the algebras N.

Let W denote the set of weights of T in N. If $\dim(T)=\dim(N/N^2)=m$ then the subset of W arising from the induced representation of T on N/N^2 has precisely m elements, say $\{\alpha_1, \cdots, \alpha_m\}$. Every $\alpha \in W$ then has a unique representation $\alpha=\sum c_i\alpha_i$ with each c_i a nonnegative integer and $c_i<p$ if the characteristic of k is $p>0$. For such an α we call the sum (in Z) $\sum c_i$ the height of α and denote it by $|\alpha|$. For α in W, denote by N_α the weight space for α in N.

DEFINITION. A T-algebra is called positive if

(i) $\dim(T)=\dim(N/N^2)$,

(ii) N is graded by the heights of the weights, i.e., if $N(j)=\bigoplus_{|\alpha|=j} N_\alpha$ then $[N(j), N(k)]\subset N(j+k)$.

REMARK. Condition (ii) is superfluous in characteristic 0. However, in characteristic $p>0$ it has such consequences as $N^r=0$ for $r>(p-1)\cdot\dim(T)$.

2. When T is a Cartan subalgebra of a complex semisimple Lie algebra G, $T+N$ a Borel subalgebra of G and W the weights of T in N, it is well known that N is the unique positive T-algebra with corresponding weight system W. This fact is a special case of the following theorem.

AMS (MOS) subject classifications (1970). Primary 18H25, 17B30; Secondary 05C99.

Key words and phrases. Lie algebra cohomology, weight systems, deformations of Lie algebras.

THEOREM 1. *Let N be a positive T-algebra and suppose that $H^2(N/N^r, N/N^r)^T=0$ for $r\geqq 4$. If N' is any other positive T-algebra inducing the same weight system W with the same height function and $\dim(N'_\alpha)\geqq\dim(N_\alpha)$ for each $\alpha\in W$, then N' is necessarily T-isomorphic to N.*

Although the task of verifying the H^2 hypothesis in Theorem 1 appears to be quite formidable, we are in fact able to check this condition for a large class of algebras by means of simple observations about W. For example, the collection of algebras described in [3, §5], which include the maximal nilpotent ideals of Borel subalgebras, satisfies the assumption. Also, we can show that the collection includes all the positive T-algebras N in characteristic $\neq 2$ which satisfy the following:

If $\alpha_1, \cdots, \alpha_m$ are the elements in W of height 1, then
(i) Every element in W is of the form $\alpha_{i_1}+\alpha_{i_2}+\cdots+\alpha_{i_s}$
(*) with $1\leqq i_1<i_2<\cdots<i_s\leqq m$.
(ii) If $\gamma,\ \gamma-\alpha_i,\ \gamma-\alpha_j\in W$ with $i\neq j$, $|\gamma|\geqq 3$ then $\gamma-\alpha_i-\alpha_j\in W$ and $\alpha_i+\alpha_j\notin W$.

3. The appearance of $H^2(N/N^r, N/N^r)^T$ in Theorem 1 is suggestive of the role that $H^2(N,N)^T$ plays in a rigidity theorem for algebras acted upon by T. The set of T multiplications in a vector space N on which T operates clearly form an algebraic subvariety of the affine space $N^*\wedge N^*\otimes N$. Thus we are led to a restricted deformation theory for such algebras. If T operates diagonally on N then $H^*(N,N)^T$ plays the role in this theory that $H^*(L,L)$ plays in the deformation theory of ordinary Lie algebras (cf. [4]). For example, we have the concept of T-rigidity and the theorem: If μ is a T-multiplication on N and if $H^2((N,\mu),(N,\mu))^T=0$ then (N,μ) is T-rigid. Also, $H^3((N,\mu),(N,\mu))^T$ arises as obstructions to integrability of 2-cocycles.

We remark that the hypothesis $H^2(N/N^r, N/N^r)^T=0$, for $r\geqq 4$, of Theorem 1 is strictly stronger for positive T-algebras than the vanishing of $H^2(N,N)^T$.

4. Suppose characteristic $(k)\neq 2$. A remarkable class of positive T-algebras over k is obtained by generalizing the Coxeter-Dynkin diagram for A_l.

Let Γ be an undirected graph [5] with at most one edge connecting any two vertices and without loops (i.e., for any vertex v, (v,v) is not an edge). A section graph g of Γ is a subgraph such that any edge in Γ which connects two vertices of g is in g. By a subtree t of Γ we shall mean a connected section graph with no circuits.

DEFINITION. A collection S of subtrees of Γ will be called admissible if
(i) whenever $t\in S$ and t' is a subtree of t then $t'\in S$.

(ii) $\Gamma = \bigcup_{t\in S} t$. (So each vertex and each edge in Γ is in some t of the collection.)

To each admissible collection S of subtrees of Γ we associate a Lie algebra N_S as follows:

Arbitrarily assign a direction $v_i \to v_j$ to each edge (v_i, v_j) in Γ. We define $\varepsilon: S\times S \to \{\pm 1, 0\}$ such that $\varepsilon_{t,t'}=0$ if either $t\cap t' \neq \varnothing$ or $t\cup t' \notin S$, otherwise $\varepsilon_{t,t'}=1$ if the unique edge connecting t to t' is directed $t\to t'$ and $\varepsilon_{t,t'}=-1$ if this edge is directed $t'\to t$. Now let N_S be the k-algebra with basis S such that $[t, t'] = \varepsilon_{t,t'} t\cup t'$.

The Jacobi identity is readily verified and so N_S is a nilpotent Lie algebra generated by the set of vertices. Next we point out that, for any vertex v of Γ, there is a unique derivation a_v of N_S such that $a_v(v)=v$ and $a_v(w)=0$ for any vertex $w\neq v$. The set of derivations a_v span a maximal diagonizable algebra T_S of derivations of N_S. One sees that N_S is a positive T_S-algebra and it is easy to verify that it satisfies property (*). Thus, by Theorem 1, N_S is the unique T_S-algebra producing its system of weights. In particular, its isomorphism class is independent of the choice of directions in Γ.

The simplest of graphs consists of a single path and if, in this case, S is taken to be the collection of all subtrees, N_S is the nilpotent subalgebra corresponding to the positive weights in an algebra of type A_l.

Thus, it is not surprising that the algebras N_S have structural and cohomological properties like those of the maximal nilpotent subalgebras of Borel subalgebras. For example, using the classical spectral sequences of Hochschild-Serre [1] as well as combinatorial properties of graphs, we can show:

THEOREM 2. *Let characteristic* $(k)=0$. *Suppose* Γ *is a graph,* S *an admissible collection of subtrees. Then* $H^i(N_S, N_S)^{T_S}=0$ *for* $i\geqq 2$.

Then, following methods used in [3], we prove:

THEOREM 3. *Let characteristic* $(k)=0$. *If* B_S *is the semidirect sum* T_S+N_S *then* $H^*(B_S, B_S)=0$.

Finally, we announce a characterization of the graph algebras by property (*). First we assign to each positive T-algebra N a graph $\Gamma(N)$ as follows: Let $\alpha_1, \cdots, \alpha_m$ be the elements in W of height 1, then $\Gamma(N)$ is the graph on m vertices $v_1, \cdots, v_m$ such that (v_i, v_j) is an edge if and only if $\alpha_i+\alpha_j$ is in W. Let S be any admissible collection of subtrees of $\Gamma(N)$. Then T acts on N_S via the isomorphism $T\to T_S$ given by $x\to \sum_i \alpha_i(x) a_{v_i}$.

THEOREM 4. *Let N be a positive T-algebra satisfying property* (*). *For each* $\alpha=\alpha_{i_1}+\cdots+\alpha_{i_s}$ *in* W, *let* g_α *be the section graph of* $\Gamma(N)$ *with vertices* $v_{i_1},\cdots,v_{i_s}$. *Then each such* g_α *is a subtree of* $\Gamma(N)$; *the set* $S=\{g_\alpha|\alpha\in W\}$ *is an admissible collection of subtrees of* $\Gamma(N)$; *and* N *is* T-*isomorphic to* N_S.

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