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A DEGREE FOR NONACYCLIC MULTIPLE-VALUED TRANSFORMATIONS

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The objective of this note is the definition of a degree generalizing that of Leray-Schauder [**1**] for certain possibly nonacyclic set-valued compact transformations of a ball in an arbitrary Banach space. The degree definition achieved extends that of [**3**] and depends on the material in [**2**], [**3**], [**4**] and [**5**]. It is new even for the finite-dimensional case.

Let E be a Banach space. Write D for the closure of a convex open set and $\dot{D}$ for its boundary. Denote by E_N an N-dimensional subspace of E and by D_N and $\dot{D}_N$ the intersections $E_N \cap D$ and $E_N \cap \dot{D}$, respectively. (We tacitly assume E is infinite dimensional but the finite-dimensional consequences amount to identification of E and some E_N.)

The symbol F refers to a transformation $F: D \to E$ which is upper semicontinuous, takes points into closed sets, and is compact in the sense that $\mathrm{cl}(F(D))$ is compact. Finally, for every closed set K, $F^{-1}(K)$ is required to be closed. Among other consequences, the graph $\Gamma(F)$ is closed where $\Gamma(F)=\{(x, y) \mid y \in F(x), x \in D\} \subset D \times C$, C compact$\}$. We write $\Gamma(F_N)$ and $\Gamma(\dot{F}_N)$ for the corresponding graphs when D is replaced by D_N or $\dot{D}_N$. A *fixed point* $\bar{x}$ of F satisfies $\bar{x} \in F(\bar{x})$.

Denote the *r-dimensional singular set* by $\sigma_r=\{x \mid H^r F(x) \neq 0\}$ where the cohomology groups are assumed to be the Alexander Spanier reduced groups with integer coefficients. The total singular set is denoted by $\sigma=\bigcup_r \sigma_r$.

Let p denote the effective bound for nonacyclicity, namely

$$p = 1 + \sup_r\{r + \dim \sigma_r \mid \sigma_r \neq \varnothing\},$$

where $\dim \sigma_r$ is the maximum covering dimension for finite covers of subsets of σ_r which are closed in D.

The transformation F is *admissible* if, besides the earlier restrictions, (a) F is fixed point free on $\dot{D}$, (b) $\sigma_r = \varnothing$ except for a finite set of indices, (c) σ_r is contained in a finite subspace E_S, and (d) for $x \in \sigma$, $H^*F(x)$ is

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finitely generated. Conditions (a)–(d) are obviously maintained when F is replaced by F_N and D_N, $\dot{D}_N$ and E_N replaces D, $\dot{D}$ and E.

Write $f=I-F$. It is easy to see that for some symmetric convex open set U about θ, $f(D)$ is disjunct from U. Hence, arguing from a partition of the identity subordinate to an open cover α of $\mathrm{cl}(F(D))$, there is an E_N depending on α for which $N \geqq \max(S, p+2)$, and $Q_N:\mathrm{cl}(F(D)) \to E_N$ for which $z=Q_N z+u$, $u \in U$, $z \in \mathrm{cl}(F(D))$.

Write

$$f_N = I_N - Q_N F_N, \qquad \dot{f}_N = f_N \mid \dot{D}_N.$$

Let p_N, q_N be the projections of $\Gamma(F_N)$ on D_N and on $F(D_N)$ and write $\dot{p}_N$, $\dot{q}_N$ when $\dot{F}_N$, $\dot{D}_N$ replace F_N, D_N. Finally define

$$\dot{T}_N = (p_N - Q_N q_N) \mid \Gamma(\dot{F}_N).$$

The induced homomorphisms on the $(N-1)$-dimensional cohomology groups are indicated as usual by an upper asterisk, thus $\dot{f}_N^*$, p_N^*, T_N^*.

LEMMA. *For an admissible F, $\dot{f}_N^* = \dot{p}_N^{*-1}\dot{T}_N^*$ is a homomorphism on $H^{N-1}(S^{N-1} \times K)$ to $H^{N-1}(\dot{D}_N)$ where K is a closed interval and $\dot{p}_N^*(N-1)$ is an isomorphism* [**4**], [**5**].

Let γ^{N-1} be a generator of $H^{N-1}(S^{N-1} \times K)$ which, by the deformation retraction induced isomorphism r^*, can be identified with a generator of $H^{N-1}(S^{N-1})$. Let γ_{N-1} be a generator of $H_{N-1}(S^{N-1})$ where the Kronecker index of γ^{N-1} and γ_{N-1} is 1.

DEFINITION. *The relative degree d'_N* is defined as

$$d'_N = \varepsilon(\cap\, \gamma_{N-1}) \dot{f}_N^* \gamma^{N-1}$$

where ε is the augmentation homomorphism $H_0(S^{N-1}) \to J$.

THEOREM. *d'_N is an integer independent of the choice of E_N, U, Q_N (subject to their defining restrictions).*

Hence our desired degree is d'_N and is henceforth denoted by $d[f]$. The degree has the following critical properties.

THEOREM. *For admissible F if $d[f] \neq 0$, F admits a fixed point.*

THEOREM. *If a homotopy h exists satisfying the same conditions as an admissible transformation and if $F=h(\ ,0)$ and $F_1=h(\ ,1)$, then* $d[1-F] = d[1-F_1]$.

THEOREM. *If F is the constant map $D \to x_0 \in D \cap \dot{D}^\sim$ then* $d[f]=1$.

The restriction to convex domains can be weakened. For instance

THEOREM. *Let A be a deformation retract of D with retracting map r.*

Suppose the interior of A is nonempty. If F is admissible on A to E then d[f] can be defined to satisfy the properties enunciated in the preceding theorems.

Details will be published elsewhere.

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