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ABSOLUTELY SUMMING, L_1 FACTORIZABLE OPERATORS AND THEIR APPLICATIONS

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Grothendieck asked in [3], Problem 2, p. 72 and in the remarks on p. 39, whether every 1-absolutely summing operator between two Banach spaces can be factored through an L_1 space. Theorem 2 announces the negative answer to this question.

Corollaries 1 and 2 provide counterexamples to two other questions equivalent to Problem 2, and mentioned in [3]: Can every operator T, whose adjoint T' is 1-absolutely summing, be factored through a $C(K)$ space? Is every operator which has the form UV, where U' and V are both 1-absolutely summing, an integral operator?

Theorems 3 and 5 establish the existence of a sequence of finite-dimensional Banach spaces which have the property that their unconditional basis constants tend to infinity. This answers the question mentioned for example in [1], [4], [5], [7] and asked also by A. Pełczyński and H. P. Rosenthal.

Part (5) of Theorem 4 settles a conjecture of McCarthy [8, p. 269] regarding the distance of $\mathscr{L}(l_2^n, l_2^n)$ from the subspaces of l_1.

Theorem 5 answers Problem 2 [6] by proving that when $1 \leqq p \neq 2 \leqq \infty$, M_{σ_p} (see definition below) has no unconditional basis.

Detailed proofs of these and other results will be given elsewhere.

Let $\mathscr{L}(E, F)$ denote the space of operators between two Banach spaces E and F. $E \otimes^\alpha F$ denotes the completion under the α norm of the algebraic tensor product $E \otimes F$. In particular, $l_2 \otimes^\vee l_2$, $l_2 \otimes^\wedge l_2$ and $l_2 \otimes^\sigma l_2$ are the spaces of *compact*, *integral* and *Hilbert-Schmidt* operators respectively, from l_2 to l_2. Here $\vee$ and $\wedge$ denote the "least" and "greatest" cross-norms respectively [3]. Other classes of operators considered here are:

(1) $\Pi_p(E, F)$ $(1 \leqq p \leqq \infty)$, the space of *p-absolutely summing* operators from E to F equipped with the norm π_p [9].

(2) $I_p(E, F)$, the space of *p-integral* operators from E to F equipped with the norm i_p [10].

(3) $\Gamma_p(E, F)$, the space of L_p*-factorizable* operators from E to F, that is, $T \in \Gamma_p(E, F)$ if and only if $T \in \mathscr{L}(E, F)$ and there is a positive measure space (Ω, Σ, μ) and $A \in \mathscr{L}(E, L_p(\mu))$, $B \in \mathscr{L}(L_p(\mu), F'')$ such that $iT = BA$,

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where $i:F \to F''$ is the canonical injection. Here $\gamma_p(T) = \inf \|A\| \|B\|$, taken over all possible (Ω, Σ, μ) and factorizations A, B, is the norm associated with $\Gamma_p(E, F)$ [2].

The *unconditional basis constant* of a Banach space E is defined as

$$\chi(E) = \inf_{\{e_i\}_{i\in I}} \sup_{\varepsilon_i = \pm 1, x_i} \Big\|\sum_{i\in I} \varepsilon_i x_i e_i\Big\| \Big/ \Big\|\sum_{i\in I} x_i e_i\Big\|$$

where the supremum is taken over all the choices of signs $\varepsilon_i = \pm 1$ with $\varepsilon_i = 1$ for all but finitely many i, and over all vectors $\sum_{i\in I} x_i e_i$ in E, and the infimum ranges over all possible unconditional bases $\{e_i\}_{i\in I}$ of E.

For any ideal norm α ([2], [11]) and a Banach space E we set $\alpha(E) = \alpha(I)$, where I is the identity operator on E. The *distance* between two isomorphic Banach spaces E and F is defined as

$$d(E, F) = \inf \|T\| \, \|T^{-1}\|,$$

taken over all isomorphisms T from E onto F.

Let M be the linear vector space of all matrices $a = (a_{ij}), i, j = 1, 2, 3, \ldots$, in which only a finite number of elements a_{ij} is not zero. For $1 \leqq p < \infty$ and $a \in M$ define the norms

$$\sigma_p(a) = [\operatorname{trace}(aa^*)^{p/2}]^{1/p},$$

and

$$\sigma_\infty(a) = \max\{\sum a_{ij} x_i y_j; \sum x_i^2 = \sum y_j^2 = 1\}.$$

Let M_{σ_p} be the completion of M under the norm σ_p [6], [8], and let $M_{\sigma_p}^n$ be the subspace consisting of all $a \in M$ for which $a_{ij} = 0$ if $\max(i, j) > n$. M_{σ_p} is reflexive if $1 < p < \infty$; $M'_{\sigma_p} = M_{\sigma_q}$ where $1/p + 1/q = 1$.

Finally, given two positive functions f and g defined on the integers, we say that $f(n) \gtrsim g(n)$ if the sequence $g(n)/f(n)$ is bounded, and if also $g(n) \gtrsim f(n)$ we then write $f(n) \sim g(n)$.

Our key to the proofs of the results mentioned here is the following theorem:

THEOREM 1. *Let J_n (respectively, I_n) be the natural inclusion of $l_2^n \otimes^\wedge l_2^n$ (respectively, $l_2^n \otimes^\vee l_2^n$) to $l_2^n \otimes^\sigma l_2^n$. Then*
(1) *$\gamma_1(J_n) \sim n$ and $\pi_1(J_n) \sim n^{1/2}$.*
(2) *$\gamma_1(I_n) \sim n^{3/2}$ and $\pi_1(I_n) \sim n$.*

Let $X^{(0)} = X$ and $X^{(i)}, i = 1, 2, 3, \ldots$, denote the ith adjoint of a Banach space X. Of course if $X = l_2 \otimes^\vee l_2$, then $X^{(1)} = l_2 \otimes^\wedge l_2, X^{(2)} = \mathscr{L}(l_2, l_2)$, $X^{(3)} = \mathscr{L}(l_2, l_2)'$ etc. Theorem 1 provides the following counterexamples to Problem 2 of [3, p. 72]:

THEOREM 2. *For each $i = 0, 1, 2, \ldots$, there is a 1-absolutely summing* *operator T_i mapping $(l_2 \otimes^{\vee} l_2)^{(i)}$ to $l_2 \otimes^{\sigma} l_2$ which does not factor through any L_1 space.*

COROLLARY 1. *For each $i = 0, 1, 2, \ldots$, there is an operator T_i from $l_2 \otimes^{\sigma} l_2$ to $(l_2 \otimes^{\vee} l_2)^{(i)}$ whose adjoint T_i' is 1-absolutely summing, and yet T_i does not factor through any $C(S)$ space.*

COROLLARY 2. *There is an operator of the form UV which is not 1-integral, and yet both U' and V are 1-absolutely summing.*

THEOREM 3. *Let $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$. Then*

$$\begin{aligned} \chi(l_{q'}^n \otimes^{\vee} l_{p'}^n) = \chi(l_p^n \otimes^{\wedge} l_q^n) &\gtrsim n^{1/2}; && \text{if } \infty \geqq q, p \geqq 2,\\ &\gtrsim n^{1-1/q}; && \text{if } p \geqq 2 \geqq q \geqq 1,\\ &\gtrsim n^{1-1/p}; && \text{if } q \geqq 2 \geqq p \geqq 1,\\ &\gtrsim n^{3/2-1/p-1/q}; && \text{if } 2 \geqq p, q \geqq 1. \end{aligned}$$

COROLLARY 3. *If $p, q > 1$, then $l_p \otimes^{\wedge} l_q$, and $l_{q'} \otimes^{\vee} l_{p'}$, are not isomorphic to any complemented subspace of a Banach space with an unconditional basis.*

REMARK. It was proved in **[6]** that if $1/p + 1/q \geqq 1$ and $1 \leqq p, q < \infty$ then $l_p \otimes^{\vee} l_q$ is not isomorphic to a subspace of a space with an unconditional basis. Our approach therefore provides the other case as well with the weaker conclusion.

COROLLARY 4. *Let $1 \leqq r, p \leqq 2 < q \leqq \infty$. Then $\Pi_{r'}(l_q, l_p)$ and $I_r(l_p, l_q)$ are not isomorphic to any complemented subspace of a Banach space with an unconditional basis.*

THEOREM 4. *Let $\alpha = \wedge$ or $\vee$, $E_\alpha = l_2^n \otimes^{\alpha} l_2^n$ and I be the identity operator on E_α. Let $\varphi: L_1(\mu) \to E_\alpha$ be any quotient map and $i: E_\alpha \to L_\infty(\mu)$ be any isometric embedding. Then*

(1) $\pi_1(E_\alpha) \sim n$.
(2) $i_\infty(E_\alpha) \sim n$.
(3) $\gamma_1(E_\alpha) \sim n$.
(4) $\gamma_2(E_\alpha) = \sqrt{n}$.
(5) $\operatorname{Inf}\{d(E_\alpha, Y); Y \subset l_1\} \sim \sqrt{n}$.
(6) $\operatorname{Inf}\{\pi_1(u); u: E_\alpha \to L_1(\mu)$ *and* $\varphi u = I\} \sim n^{3/2}$.
(7) $\operatorname{Inf}\{\|u\|; u: L_\infty(\mu) \to L_1(\mu)$ *and* $\varphi u i = I\} \sim n^{3/2}$.
(8) $\pi_1(\varphi) \sim \sqrt{n}$.
(9) $\operatorname{Inf}\{d(E_\wedge, Y); Y \subset l_p\} \sim \sqrt{n}$, *if* $1 \leqq p < \infty$.

THEOREM 5. *For any $1 \leqq p \leqq \infty$, $\chi(M_{\sigma_p}^n) \sim n^{|1/p-1/2|}$. If $p \neq 2$, M_{σ_p} is not isomorphic to a complemented subspace of a Banach space with an unconditional basis.*

REMARKS. It is possible to obtain stronger results by adding the following definition: A Banach space E is a U_λ-space ($\lambda \geqq 1$) if given any finite-dimensional subspace $G \subseteq E$, there is a closed subspace F, $G \subseteq F \subseteq E$, a space U with an unconditional basis, and operators $S \in \mathscr{L}(F, U)$, $T \in \mathscr{L}(U, F)$ such that TS is the identity on F and $\|S\| \, \|T\| \, \chi(U) \leqq \lambda$. Denote

$$\chi_u(E) = \inf\{\lambda; E \text{ is a } U_\lambda\text{-space}\}.$$

Observe that in the definition U and F may vary with each choice of G and may be finite- or infinite-dimensional. All $\mathscr{L}_p$ spaces are U_λ-spaces for appropriate λ's. Also $\chi(E) \geqq \chi_u(E)$ for every E. We now have

THEOREM 6. *If* $A \in \Pi_1(E, F)$, *then* $\gamma_1(A) \leqq \pi_1(A)\chi_u(E)$.

Our results imply that none of the spaces $(l_p \otimes^\vee l_q)^{(i)}$ $(1 \leqq p, q < \infty$, $i = 0, 1, 2, \ldots)$, M_{σ_p} $(1 \leqq p \neq 2 \leqq \infty)$ are U_λ-spaces for any λ, since on each of these spaces there is a 1-absolutely summing operator which does not factor through any L_1 space. Moreover, the estimates of Theorems 3 and 5 remain the same if χ is replaced by χ_u.

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