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INDEX THEORY FOR SINGULAR QUADRATIC FUNCTIONALS IN THE CALCULUS OF VARIATIONS[1]

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1. **Introduction.** Let P, Q, and R be real-valued $n \times n$ matrix functions defined on the interval $[a, b)$. Assume that P, Q, and R are continuous on $[a, b)$ and that $P(t)$ and $R(t)$ are symmetric matrices for each t in $[a, b)$. We do not assume that Q is symmetric. Also assume that R has the property that its value for any t in $[a, b)$ is positive definite, that is, $v^*R(t)v > 0$ for all n-vectors $v \neq 0$ and for each t in $[a, b)$. Let

$$J(x, y)\Big|_{e_1}^{e_2} = \int_{e_1}^{e_2} [\dot{x}^*(t)R(t)\dot{y}(t) + x^*(t)Q(t)\dot{y}(t) + \dot{x}^*(t)Q^*(t)y(t) + x^*(t)R(t)y(t)]\, dt \qquad (a \leqq e_1 \leqq e_2 < b), \tag{1.1}$$

for x and y in the class A of vector-valued functions described below. Also let

$$J_e(x, y) = J(x, y)\Big|_a^e, \qquad J_e(x) = J_e(x, x), \tag{1.2}$$

$$J(x, y) = \liminf_{e \to b-} J_e(x, y), \qquad J(x) = \liminf_{e \to b-} J_e(x) \tag{1.3}$$

for x and y in A. The class A is the set of vector-valued functions $x^*(t) = (x_1(t), \ldots, x_n(t))$, $a \leqq t \leqq b$, satisfying

(i) $x(t)$ is continuous on the interval $[a, b]$ and $x(a) = x(b) = 0$,

(ii) $x(t)$ is absolutely continuous and $\dot{x}^*(t)\dot{x}(t)$ is Lebesgue integrable on each closed subinterval of $[a, b)$. A is a vector space of functions.

J is said to be *singular* at a point t in $[a, b]$ if the determinant of $R(t)$ is zero or not defined. The point $t = b$ is a singular point in this paper.

2. **Preliminaries.** What is presented here is part of a quadratic form theory developed and used extensively by Hestenes [3], [4]. Let $Q(x)$

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be a quadratic functional defined on a vector space V and let $Q(x, y)$ be its associated symmetric bilinear functional. Two vectors x and y in V are said to be *Q-orthogonal* whenever $Q(x, y) = 0$. A vector x is said to be *Q-orthogonal* to a subset S of V whenever $Q(x, y) = 0$ for every y in S. By the *Q-orthogonal complement* S^Q of the set S in V is meant the set of all vectors x in V that are Q-orthogonal to S. S^Q is a subspace of V. A vector in S that is Q-orthogonal to S is called a *Q-null vector* of S. The intersection $S \cap S^Q$ is the set of Q-null vectors of S and is usually denoted by S_0. If S is a subspace of V, then so is S_0.

Let S be any subspace in V. We define the *nullity* $n(S)$ *of* Q *on* S or more simply the *Q-nullity* of S to be the dimension of the subspace $S_0 = S \cap S^Q$ of Q-null vectors in S. We define the *signature* $s(S)$ *of* Q *on* S, the *index of* Q *on* S, or the *Q-signature of* S to be the dimension of a maximal subspace M of S on which $Q < 0$ if this dimension is finite. If no such finite dimensional space exists, we set $s(S) = \infty$. By $Q < 0$ on M we mean that $Q(x) < 0$ for each nonzero x in M. It turns out that the dimension $s(S)$ of M is independent of the choice of M so that the notion of signature is well defined.

THEOREM 2.1. *If the Q-signature of S is finite where S is a subspace of V, then it is given by one of the following quantities*:

(i) *the dimension of a maximal subspace* M *in* S *on which* $Q < 0$;

(ii) *the dimension of a maximal subspace* M *of* S *on which* $Q \leqq 0$ *and having* $M \cap S_0 = 0$;

(iii) *the dimension of a minimal subspace* M *of* S *such that* $Q \geqq 0$ *on* $S \cap M^Q$;

(iv) *the least integer* k *such that there exist* k *linear functionals* $L_1, \ldots, L_k$ *on* S *with the property that* $Q(x) \geqq 0$ *for all* x *in* S *satisfying the conditions* $L_\alpha(x) = 0$ $(\alpha = 1, 2, \ldots, k)$.

3. **Results.** The main purpose of this paper is to announce the results presented in this section. The details and more results are to appear elsewhere.

The definition of a singular conjugate point is found in Tomastik [7, p. 61] and Chellevold [1, p. 333]. It extends the definition of Morse and Leighton [5, p. 253], who treated the case $n = 1$. For $a \leqq e \leqq b$ let $A(e) = \{x \in A : x(t) = 0 \text{ for } e \leqq t \leqq b\}$, where A is defined in §1 of this paper. Define the set B in A to be the union of the sets $A(e)$ for $a < e < b$. Observe that B is actually a subspace of A.

THEOREM 3.1. *The following conditions are equivalent for some non-negative integer* k:

(i) *The signature of* J *given by* (1.3) *on* B *is* k.

(ii) *There is an* ε_0 *in* (a, b) *such that* $\varepsilon_0 \leqq \varepsilon < b$ *implies that the signature of* J *given by* (1.3) *on* $A(\varepsilon)$ *is* k.

(iii) *The point* a *has exactly a finite number* k *of nonsingular conjugate points on* $a < t < b$.

(iv) *The point* b *has exactly a finite number* k *of singular conjugate points on* $a < t < b$.

(v) b *is not conjugate to* b.

Theorem 3.1 above contains Theorem 4.4, p. 337, of Chellevold [**1**]. Let $U(t)$ be a conjugate system satisfying Euler's equation

$$[R(t)\dot{U}(t) + Q^*(t)U(t)]' = [Q(t)\dot{U}(t) + P(t)U(t)] \tag{3.1}$$

and the conditions $U(a) = 0$, $\dot{U}(a) = I$, $\det U(t) \neq 0$ for t near b. Let us remark that there are J's which do not possess such conjugate systems. For y in A and for t near b set

$$S[y(t), a] = y^*(t)\{[R(t)\dot{U}(t) + Q^*(t)U(t)]U^{-1}(t)\}y(t). \tag{3.2}$$

Let D be a subspace in A satisfying $B \subseteq D \subseteq A$. The condition that $\liminf_{t\to b-} S[y(t), a] \geqq 0$ for each y in D satisfying $\liminf_{t\to b-} J(y)\,|_a^t < \infty$ is called the *singularity condition relative to D and belonging to* $[a, b]$.

THEOREM 3.2. *Assume that* $s(B)$ *is finite. Let* D *be any subspace with* $B \subseteq D \subseteq A$. *Let* C *be a subspace in* B *maximal relative to having* $J < 0$. *Let* $C^J = \{x \in A : J(x, y) = 0$ *for all* y *in* $C\}$. *The following conditions are equivalent*:

(i) *If* x *is in* $D \cap C^J$, *then* $J(x) < \infty$ *implies* $\liminf_{e\to b-} S[x(e), a] \geqq 0$.

(ii) *If* x *is in* $D \cap C^J$, *then* $J(x) \geqq 0$.

(iii) *The singularity condition relative to* D *holds*; *that is, if* x *is in* D, *then* $J(x) < \infty$ *implies* $\liminf_{e\to b-} S[x(e), a] \geqq 0$.

THEOREM 3.3. *Suppose that* $J(x, y) = \liminf_{e\to b-} J_e(x, y)$ *is bilinear on the subspace* D *where* $B \subseteq D \subseteq A$. *Assume that* $s(B)$ *is finite. Let* C *be a subspace in* B *maximal relative to having* $J < 0$. *Let* $C^J = \{x \in A : J(x, y) = 0$ *for all* y *in* $C\}$. *Then* $s(D) = s(B)$ *if and only if* x *in* $C^J \cap D$ *implies* $J(x) \geqq 0$.

COROLLARY. *If* J *is bilinear on the subspace* D *with* $B \subseteq D \subseteq A$ *and* $s(B)$ *is finite, then* $s(D) = s(B)$ *if and only if the singularity condition relative to* D *and belonging to* $[a, b]$ *holds*.

The next theorem generalizes Theorems 2.3, 4.1, and 5.1 of Tomastik [7].

THEOREM 3.4. *There is a subspace* C *of finite dimension* k *in* B *with* C *maximal relative to having* $J < 0$ *and* $J \geqq 0$ *on* $C^J \cap D$ *holds for a subspace*

D with $B \subseteq D \subseteq A$ if and only if there are k conjugate points to b in $(a, b]$ and the singularity condition relative to D and belonging to $[a, b]$ is satisfied.

COROLLARY. *There is a subspace C of finite dimension k in B with C maximal relative to having $J < 0$ and $J \geqq 0$ on C^J holds if and only if there are k conjugate points to b in $(a, b]$ and the singularity condition relative to A and belonging to $[a, b]$ is satisfied.*

COROLLARY. *For any subspace D with $B \subseteq D \subseteq A$, $J \geqq 0$ on D holds if and only if there are no conjugate points to b in $(a, b]$ and the singularity condition relative to D and belonging to $[a, b]$ is satisfied.*

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