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SMOOTH S^1 ACTIONS AND BILINEAR FORMS[1]

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Let S^1 denote the multiplicative group of complex numbers of norm 1. Let X denote a smooth S^1 manifold, i.e., X consists of an underlying smooth manifold denoted by $|X|$ together with a smooth action of S^1. The equivariant complex K theory of X is $K^*_{S^1}(X) = K^0_{S^1}(X) \oplus K^1_{S^1}(X)$. It is a module over $R(S^1)$ the complex representation ring of S^1. This is the ring $Z[t, t^{-1}]$. For our purposes there are two important sets of prime ideals in $Z[t, t^{-1}]$:

(i) the set P_1 consisting of the principal ideals of the form $\mathfrak{p} = (\Phi_{p^r}(t))$ generated by the cyclotomic polynomial $\Phi_{p^r}(t)$ associated to the prime power p^r, i.e., $P_1 = \{(\Phi_{p^r}(t)) \mid \forall \text{ primes } p \text{ and integers } r\}$.

(ii) the set $P = \{(\Phi_m(t)) \mid \forall \text{ positive integers } m\}$.

The localized ring $R(S^1)_P$ is denoted by R. It is the subring of the field of fractions of $R(S^1)$ consisting of fractions a/b with b prime to all the ideals of P. Let $K^*_{S^1}(X)_P = K^*_{S^1}(X) \otimes_{R(S^1)} R$. The Atiyah-Singer index homomorphism [1] $\mathrm{Id}^X_{S^1}: K^0_{S^1}(TX) \to R(S^1)$ induces a homomorphism

$$\mathrm{Id}^X: K^0_{S^1}(TX)_P \to R.$$

Here TX is the tangent bundle of X and $|X|$ is compact without boundary. Suppose that $|X|$ is a spin^c manifold. Then there is an isomorphism

$$K^*_{S^1}(X)_P \xrightarrow{\Delta^X} K^*_{S^1}(TX)_P$$

of R modules [6] and we can define an R valued bilinear form $\langle\ \rangle_X$ on $K_{S^1}(X)_p$ by

$$\langle a, b\rangle_X = \mathrm{Id}^X(\Delta^X(a) \cdot b).$$

THEOREM 1 [2]. *The bilinear form $\langle\ \rangle_X$ is nonsingular, i.e., the associated homomorphism*

$$K^*_{S^1}(X)_P \xrightarrow{\Phi^X} \mathrm{Hom}_R(K^*_{S^1}(X)_P, R)$$

is surjective where $\Phi^X(a)[b] = \langle a, b\rangle_X$.

This result was conjectured in a similar form in [6].

A useful consequence of Theorem 1 is this: Set $\bar{K}^*_{S^1}(X) = K^*_{S^1}(X)_P/T_X$ where T_X denotes the R torsion subgroup of $K^*_{S^1}(X)_P$. The bilinear form

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$\langle\ \rangle_X$ defines a bilinear form again denoted by $\langle\ \rangle_X$ on $\tilde{K}^*_{S^1}(X)$.

THEOREM 1′. *The associated homomorphism* $\tilde{K}^*_{S^1}(X) \to \mathrm{Hom}_R(\tilde{K}^*_{S^1}(X), R)$ *is an isomorphism.*

Before mentioning applications, let me discuss the problems to which we wish to apply this result.

(i) If S^1 acts effectively on M and if N is homotopy equivalent to M, does S^1 act effectively on N?

(ii) If S^1 acts effectively on a smooth manifold, what are the relations among the representations of S^1 on the tangent spaces at the points fixed by S^1 and the global invariants of the manifold, e.g., its Pontryagin classes and its cohomology?

Towards answering these questions, we introduce the set $S_{S^1}(Y)$ associated to the closed S^1 manifold Y. It consists of equivalence classes of pairs (X, f) where $f: X \to Y$ is an equivariant map such that

(1) $|f|: |X| \to |Y|$ is a homotopy equivalence;

(2) $|f^{S^1}|: |X^{S^1}| \to |Y^{S^1}|$ is a homotopy equivalence.

Two pairs (X_i, f_i), $i = 0, 1$, are equivalent if there is an S^1 *homotopy equivalence* $\phi: X_0 \to X_1$ such that $f_1\phi$ is S^1 homotopic to f_0. The equivalence class of (X, f) is denoted by $[X, f]$.

Suppose that $|Y|$ is a spin^c manifold. Then if $[X, f] \in S_{S^1}(Y)$, $|X|$ is a spin^c manifold and we can define an induction homomorphism [**3**]

$$f_*: \tilde{K}^*_{S^1}(X) \to \tilde{K}^*_{S^1}(Y)$$

by $\langle f_*(x), y\rangle_Y = \langle x, f^*(y)\rangle_X$. If 1_X denotes the identity of the algebra $\tilde{K}^*_{S^1}(X)$, then the element $f_*(1_X)$ is a very important geometric invariant of the situation. It relates the algebra $\tilde{K}^*_{S^1}(Y)$ with the differential structures on $|X|$ and $|Y|$ and with the representations of S^1 on the normal bundles to the fixed sets $X^{S^1} \subset X$ and $Y^{S^1} \subset Y$. In order to illustrate these relations in a simple manner, we restrict ourselves to the *case where* Y^{S^1} *consists of isolated points. In addition, we want to assume that the odd dimensional rational cohomology of* $|Y|$ *vanishes.* In this situation the natural homomorphism $K^*_{S^1}(Y) \to K^*(|Y|)$ induces a homomorphism $\tilde{K}^*_{S^1}(Y) \to K^*(|Y|) \otimes Q$ and the composition with the Chern character isomorphism ch to $H^*(|Y|, Q)$ is denoted by ϕ_Y. If $p \in Y^{S^1}$, the representation of S^1 on the normal bundle of Y^{S^1} at p is denoted by NY_p. We may assume it to be a complex representation of S^1.

We remark that if $[X, f] \in S_{S^1}(Y)$, $f^{S^1}: X^{S^1} \to Y^{S^1}$ is a homeomorphism when Y^{S^1} consists of isolated points. Let $g: |Y| \to |X|$ be a homotopy inverse to $|f|$. We can now illustrate the geometric importance of $f_*(1_X)$ and its relation with the algebra $\tilde{K}^*_{S^1}(Y)$.

THEOREM 2 [5]. *Let* $[X, f] \in S_{S^1}(Y)$. *Then* $\phi_Y(f_*(1_X)) = g^*A(|X|)/A(|Y|)$ *where* $A(|X|)$ *denotes the cohomology class associated to the tangent bundle of* $|X|$ *by the power series* $(x/2)/\sinh x/2$.

THEOREM 3 [5]. *Let* $q \in X^{S^1}$. *The restriction of* $f_*(1_X)$ *to* $p \in Y^{S^1}$ *is denoted by*

$$f_*(1_X)_p \in K^*_{S^1}(p) = R \quad \text{and} \quad f_*(1_X)_{f(q)} = \pm t^{N_q} \cdot \lambda_{-1}(NY_{f(q)})/\lambda_{-1}(NX_q) \in R.$$

Here N_q *is an integer and e.g.,* $\lambda_{-1}(NX_q) = \sum (-1)^i \lambda^i(NX_q) \in R$.

THEOREM 4 [5]. $f_*(1_X)_{f(q)}$ *is a unit of* R_{P_1}. (*Compare* [6, p. 139, *Theorem* 2.6].)

THEOREM 5 [5]. *If* $f^*: \tilde{K}^*_{S^1}(Y) \to \tilde{K}^*_{S^1}(X)$ *is an isomorphism,* $f_*(1_X)$ *is a unit of* $\tilde{K}^*_{S^1}(Y)$ *and* $f_*(1_X)_q = \pm 1 \in R$ *for all* $q \in X^{S^1}$ *and* $\phi_Y f_*(1_X) = 1 \in H^*(|Y|, Q)$.

Briefly, Theorem 2 relates $f_*(1_X)$ and Pontryagin classes, Theorem 3 relates $f_*(1_X)$ and normal representations and Theorems 4 and 5 relate $f_*(1_X)$ with the algebra $\tilde{K}^*_{S^1}(Y)$. We remark that Theorem 5 together with Theorem 2 actually implies that if f is an S^1 homotopy equivalence, $|f|$ preserves Pontryagin classes.

Here is an interesting example to illustrate the ideas. Let p, q be relatively prime integers. Choose integers a, b such that $-ap + bq = 1$. Let $N = t^p + t^q$ and $M = t^1 + t^{pq}$ denote the indicated complex 2 dimensional representations of S^1. The one point compactifications N^+ and M^+ are smooth S^1 manifolds with $|N^+| = |M^+| = S^4$. The map $\Phi: N \to M$ defined by $\Phi(z_0, z_1) = (\bar{z}_0^a z_1^b, z_0^q + z_1^p)$ is proper, hence defines a map $\Phi^+: N^+ \to M^+$ and $[N^+, \Phi^+] \in S_{S^1}(M^+)$. The invariant $(\Phi^+)_*(1_{N^+})$ is

$$(1 - t)(1 - t^{pq})/(1 - t^p)(1 - t^q) \cdot 1_{M^+} \in K^*_{S^1}(M^+).$$

For deeper applications of ideas, see [4] and [5].

Theorems 3 and 4 combine to give a comparison of the representations NX_q and $NY_{f(q)}$ as follows: Let F denote the field of fractions of $Z[t, t^{-1}]$. For each prime ideal $\mathfrak{p}$ of P let $\| \ \|_{\mathfrak{p}}$ denote the valuation defined by $\mathfrak{p}$. We interpret this as a norm on F. Then

THEOREM 6. *If* $[X, f] \in S_{S^1}(Y)$ *and* $q \in X^{S^1}$,

$$\|\lambda_{-1}(NX_{f(q)})/\lambda_{-1}(NX_q)\|_{\mathfrak{p}} = 1 \quad \textit{for all } \mathfrak{p} \in P_1.$$

REMARK. If this is true for all $\mathfrak{p} \in P$, then the real representations of $NY_{f(q)}$ and NX_q are equal.

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