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HOLOMORPHIC FOCK REPRESENTATIONS AND PARTIAL DIFFERENTIAL EQUATIONS ON COUNTABLY HILBERT SPACES

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1. **Introduction.** Differential operators on the holomorphic Fock space $F(E)$ of entire functions of Hilbert-Schmidt type on a Hilbert space E of [**Be**], [**D1**] and [**R**] describe observables of quantum systems with an infinite number of degrees of freedom [**Be**]. These operators are in general unbounded on $F(E)$, leading to the introduction of weighted holomorphic Fock spaces and their projective and inductive limits, on which suitable differential operators are bounded [**D1**], [**D2**] and [**R**]. For certain weights, existence theorems also hold [**D1**]. However, the weighted spaces are not nuclear on infinite-dimensional Hilbert space domains, so kernel representations do not in general hold. In this note we introduce instead holomorphic Fock spaces on countably Hilbert spaces and on their duals (Theorems 2.2 and 2.3), with suitably defined Hilbert-Schmidt polynomial derivatives (Theorem 2.1), again obtaining boundedness and existence theorems (Theorems 3.1 and 3.2), as well as nuclearity under certain conditions (Proposition 2.1). Moreover, these function spaces provide representations of tempered distributions in infinite dimension in the sense of [**KMP**].

2. **Holomorphic mappings on countably Hilbert spaces.** Let E be a projective limit of Hilbert spaces E_r with injective and dense linear maps $E_s \to E_r$ for $r \leqq s$ (real or integer indices), and E' its strong dual. By $P_H(^nE_r)$ and similarly on E_r', we mean the spaces of n-homogeneous Hilbert-Schmidt polynomials as in [**D1**], [**D2**]. We recall that $P_H(^nE_r')$ and $P_H(^nE_r)$ are in duality for a bilinear form $\langle\ ,\ \rangle_n$ such that $\langle x^n, x'^n\rangle_n = \langle x, x'\rangle^n$. The Hilbert-Schmidt norm on $P_H(^nE_r')$ is denoted by $\|\ \ \|_{r,n}$ and on $P_H(^nE_r)$ by $\|\ \ \|'_{r,n}$. We define the spaces of n-homogeneous Hilbert-Schmidt polynomials on E' and E to be respectively the projective and inductive limits $P_H(^nE') = \lim\text{inv}_r\, P_H(^nE_r')$ and $P_H(^nE) = \lim\text{dir}_r\, P_H(^nE_r)$, for the canonical extension and restriction maps. By $F(E_r')$ we mean the Hilbert space of entire functions f of Hilbert-Schmidt type on E_r' such that

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$\sum_{n=0}^{\infty} (1/n!) \|\hat{d}^n f(0)\|_{r,n}^2 < \infty$, where $\hat{d}^n f(x')$ denotes the nth polynomial derivative of f at x'. $\boldsymbol{F}(E_r)$ is similarly defined [**D1**], [**D2**] and [**R**]. By holomorphic Fock space on E we mean the space $\boldsymbol{F}(E)$ of all maps $f' : E \to \boldsymbol{C}$ which are restrictions to E of maps belonging to at least one of the spaces $\boldsymbol{F}(E_r)$. We equip $\boldsymbol{F}(E)$ with the locally convex inductive limit topology determined by the maps $\boldsymbol{F}(E_r) \to \boldsymbol{F}(E)$ (restrictions). By holomorphic Fock space on E' we mean the space $\boldsymbol{F}_*(E')$ of maps $f : E' \to \boldsymbol{C}$ whose restrictions to each E'_r (by the injections $E'_r \to E'$) are in $\boldsymbol{F}(E'_r)$. By $\boldsymbol{F}(E')$ we denote the space of entire functions belonging to $\boldsymbol{F}_*(E')$ (functions which are local uniform limits of their Taylor series). We equip $\boldsymbol{F}_*(E')$ and $\boldsymbol{F}(E')$ with the projective limit topology determined by the maps $\boldsymbol{F}_*(E') \to \boldsymbol{F}(E'_r)$ in the definition of $\boldsymbol{F}_*(E')$.

THEOREM 2.1. (i) *There is a unique topological isomorphism* $J : \boldsymbol{P}_H({}^nE')' \to \boldsymbol{P}_H({}^nE)$ *such that* $JT(x) = T(x^n)$ (*where* $x^n = \langle x, \ \rangle^n$) *for* T *in* $\boldsymbol{P}_H({}^nE')'$ *and* x *in* E.

(ii) *The isomorphism* (i) *is given by a hypocontinuous bilinear form* $\langle \ , \ \rangle_n : \boldsymbol{P}_H({}^nE') \times \boldsymbol{P}_H({}^nE) \to \boldsymbol{C}$ *such that* $\langle P, x'^n \rangle_n = P(x')$ *and* $\langle x^n, P' \rangle_n = P'(x)$, *for* P *in* $\boldsymbol{P}_H({}^nE')$ *and* P' *in* $\boldsymbol{P}_H({}^nE)$.

(iii) *The maps* $\boldsymbol{P}_H({}^nE') \to \boldsymbol{P}_H({}^nE'_s) \to \boldsymbol{P}_H({}^nE'_r)$ *and* $\boldsymbol{P}_H({}^nE_r) \to \boldsymbol{P}_H({}^nE_s) \to \boldsymbol{P}_H({}^nE)$ *for* $r \leqq s$ *are continuous linear injections, as well as with dense images. Moreover, the norms of the maps* $\boldsymbol{P}_H({}^nE'_s) \to \boldsymbol{P}_H({}^nE'_r)$ *and* $\boldsymbol{P}_H({}^nE_r) \to \boldsymbol{P}_H({}^nE_s)$ *are bounded above by* $K^{n(s-r)}$ *when those of the maps* $E_s \to E_r$ *are bounded above by* $K > 0$.

(iv) $\boldsymbol{P}_H({}^nE')$ *is a countably Hilbert space of continuous polynomials on* E', *and* $\boldsymbol{P}_H({}^nE)$ *a Hausdorff, complete, reflexive* (*DF*)-*space of continuous polynomials on* E.

(v) *If for each* r *the maps* $E_s \to E_r$ *for some* $s > r$ *are Hilbert-Schmidt operators* (*e.g.* E *nuclear*) *then* $\boldsymbol{P}_H({}^nE')$ *and* $\boldsymbol{P}_H({}^nE)$ *are nuclear. Moreover, all continuous* n-*homogeneous polynomials on* E *or* E' *are then of Hilbert-Schmidt type in the sense above.*

(vi) *If* (v) *holds then the topologies of* $\boldsymbol{P}_H({}^nE')$ *and* $\boldsymbol{P}_H({}^nE)$ *coincide with the topologies of uniform convergence on bounded sets of* E' *and of* E *respectively.*

The above results are obtained through a representation of $\boldsymbol{P}_H({}^nE')$ and $\boldsymbol{P}_H({}^nE)$ by tensors in lim inv$_r$ $E_r^{\vee n}$ and lim dir$_r$ $E_r'^{\vee n}$ (completed symmetric products for the Hilbert-Schmidt norms, of the spaces E_r and E'_r respectively). We observe that the continuity of the polynomials on E' (not a Fréchet nor in general a Baire space) is obtained without the assumption of (v), in contrast with the possible noncontinuity of separately continuous multilinear forms in general.

THEOREM 2.2. (i) *f' is in $\boldsymbol{F}(E)$ iff f' is an entire function with $\hat{d}^n f'(0)$ in $\boldsymbol{P}_H(^nE)$ for all n and $|||f'|||_r'^2 = \sum_{n=0}^{\infty} (1/n!) \, \|\hat{d}^n f'(0)\|_{r,n}'^2 < \infty$ for some r.*

(ii) *$\boldsymbol{F}(E)$ is a bornological (DF)-space, and Taylor series converge in the topology of $\boldsymbol{F}(E)$.*

(iii) *f is in $\boldsymbol{F}_*(E')$ iff f is the pointwise limit of a series $\sum_{n=0}^{\infty} (1/n!)P_n$ with P_n in $\boldsymbol{P}_H(^nE')$ such that $|||f|||_r^2 = \sum_{n=0}^{\infty} (1/n!) \, \|\hat{d}^n f(0)\|_{r,n}^2 < \infty$ for every r.*

(iv) *$\boldsymbol{F}_*(E')$ is a countably Hilbert space and the power series of* (iii) *(Taylor series at zero in $\boldsymbol{F}(E')$) converge to the corresponding functions in the topology of $\boldsymbol{F}_*(E')$: it follows that $\boldsymbol{F}(E')$ is dense in $\boldsymbol{F}_*(E')$.*

(v) *Power series developments in $\boldsymbol{F}(E)$ and $\boldsymbol{F}_*(E')$ converge uniformly on bounded sets.*

(vi) *If for each r the map $E_s \to E_r$ is compact for some $s > r$ then $\boldsymbol{F}_*(E') = \boldsymbol{F}(E')$.*

Proofs of (i)–(v) are by reduction to the Hilbert space case of [**D1**], [**D2**]. For (vi) local boundedness on E' is derived from local boundedness on each E_r' (cf. [**P**]).

Given T in $\boldsymbol{F}_*(E')'$ and x in E let $BT(x) = T(e^x)$, where $e^x = \exp \circ \langle x, \ \rangle$, defining the Borel transformation B on $\boldsymbol{F}_*(E')'$. Given T' in $\boldsymbol{F}(E)'$ and x' in E' let $B'T'(x') = T'(e^{x'})$, where $e^{x'} = \exp \circ x'$, defining the Borel transformation B' on $\boldsymbol{F}(E)'$. From [**D1**], [**D2**], by reduction to Hilbert space domains we get:

THEOREM 2.3. (i) *B is a topological isomorphism from $\boldsymbol{F}_*(E')'$ onto $\boldsymbol{F}(E)$.*

(ii) *B' is a topological isomorphism from $\boldsymbol{F}(E)'$ onto $\boldsymbol{F}_*(E')$.*

(iii) *There is a unique hypocontinuous bilinear form $\langle\!\langle \ , \ \rangle\!\rangle : \boldsymbol{F}_*(E') \times \boldsymbol{F}(E) \to \boldsymbol{C}$ placing $\boldsymbol{F}_*(E')$ and $\boldsymbol{F}(E)$ in separating duality, characterized by $\langle\!\langle e^x, f' \rangle\!\rangle = f'(x)$ and $\langle\!\langle f, e^{x'} \rangle\!\rangle = f(x')$.*

COROLLARY. *$\boldsymbol{F}(E)$ is a Hausdorff, complete and reflexive space, and its bounded sets are those bounded in some $\boldsymbol{F}(E_r)$.*

PROPOSITION 2.1. *If the canonical maps $E_{r+1} \to E_r$ are Hilbert-Schmidt operators with Hilbert-Schmidt norms bounded by $K \leqq e^{-1}$ then $\boldsymbol{F}_*(E') = \boldsymbol{F}(E')$ and $\boldsymbol{F}(E)$ are nuclear. Conversely, if E' is not nuclear then neither are $\boldsymbol{F}_*(E')$, $\boldsymbol{F}(E')$ nor $\boldsymbol{F}(E)$.*

3. **Partial differential equations in holomorphic Fock spaces.**

PROPOSITION 3.1. *If f (resp. f') is in $\boldsymbol{F}(E')$ (resp. $\boldsymbol{F}(E)$) then each $\hat{d}^n f(x')$ (resp. $\hat{d}^n f'(x)$) is in $\boldsymbol{P}_H(^nE')$ (resp. $\boldsymbol{P}_H(^nE)$) for all x in E and x' in E'.*

Given $P' = \sum_{n=0}^{m} P_n'$ with P_n' in $\boldsymbol{P}_H(^nE)$, the differential operators

$P'_n(d)$ may then be defined at f in $F(E')$ and x' in E' by $P'_n(d)f(x') = \langle \hat{d}^n f(x'), P'_n\rangle_n$, and $P'(d)$ by $\sum_{n=0}^m P'_n(d)$. After continuity is proved, these operators may be uniquely extended to $\boldsymbol{F}_*(E')$. Given $P = \sum_{n=0}^m P_n$ with P_n in $\boldsymbol{P}_H({}^nE')$, the differential operators $P_n(d)$ may also be defined at f' in $\boldsymbol{F}(E)$ and x in E by $P_n(d)f'(x) = \langle P_n, d^n f'(x)\rangle_n$, and $P(d)$ by $\sum_{n=0}^m P_n(d)$. Let $\boldsymbol{P}_H(E')$ denote the space of all Hilbert-Schmidt polynomials on E', that is, sums of homogeneous Hilbert-Schmidt polynomials, the same for $\boldsymbol{P}_H(E)$, $\boldsymbol{P}_H(E')\cdot \boldsymbol{F}_*(E')$ the image of $\boldsymbol{F}_*(E')$ under multiplication by Hilbert-Schmidt polynomials, and $\boldsymbol{P}_H(E)(d)\boldsymbol{F}_*(E')$ the image of $\boldsymbol{F}_*(E')$ under the action of the corresponding differential operators, with similar definitions over $\boldsymbol{F}(E)$.

THEOREM 3.1. *If the norms of the maps $E_{r+1} \to E_r$ are $K \leqq 2^{-1/2}$, the following inclusions hold, and the corresponding bilinear maps are hypocontinuous (continuous when the domain is E'):*

$$\boldsymbol{P}_H(E')\cdot \boldsymbol{F}_*(E') \subset \boldsymbol{F}_*(E'); \qquad \boldsymbol{P}_H(E)(d)\boldsymbol{F}_*(E') \subset \boldsymbol{F}_*(E'),$$
$$\boldsymbol{P}_H(E)\boldsymbol{F}(E) \subset \boldsymbol{F}(E), \quad \text{and} \quad \boldsymbol{P}_H(E')(d)\boldsymbol{F}(E) \subset \boldsymbol{F}(E),$$

the same holding with $\boldsymbol{F}_*(E')$ replaced by $\boldsymbol{F}(E')$.

The proof uses the estimates in the next lemma and, for differential mappings into $\boldsymbol{F}(E')$, beginning with operators of finite type (given by polynomials which are finite sums of powers of continuous linear forms) and extending by continuity.

LEMMA 3.1. *Given P_m in $\boldsymbol{P}_H({}^mE')$, f in $\boldsymbol{F}_*(E')$, P'_m in $\boldsymbol{P}_H({}^mE_s)$ and f' in $\boldsymbol{F}(E_r)$, with $s \geqq r$, $t \geqq \max(r, s-1)$ we have*

$$|||P_m \cdot f|||_r^2 \leqq K_m \, \|P_m\|_{r,m}^2 \, |||f|||_{r+1}^2,$$
$$|||P'_m(d)f|||_r^2 \leqq K_m \, \|P'_m\|_{s,m}'^2 \, |||f|||_{s+1}^2,$$
$$|||P'_m \cdot f'|||_{t+1}'^2 \leqq K_m \, \|P'_m\|_{t,m}'^2 \, |||f'|||_t'^2,$$
$$|||P_m(d)f'|||_{r+1}'^2 \leqq K_m \, \|P_m\|_{r+1,m}^2 \, |||f'|||_r'^2$$

where K_m is equal to 1 if $m = 1$, to 2 if $m = 2$ and to $(2m-3)!/(m-2)!$ if $m > 2$.

THEOREM 3.2. *Given P' in $\boldsymbol{P}_H(E)$ and P in $\boldsymbol{P}_H(E')$ we have*: (i) $P'(d)\boldsymbol{F}_*(E') = \boldsymbol{F}_*(E')$, (ii) $P(d)\boldsymbol{F}(E) = \boldsymbol{F}(E)$.

The proof requires the following lemmas (cf. [**D1**, Lemma 2] and [**D2**, Lemma 4.1]).

LEMMA 3.2. *Given P in $\boldsymbol{P}_H(E')$, P' in $\boldsymbol{P}_H(E)$, f in $\boldsymbol{F}_*(E')$ and f' in $\boldsymbol{F}(E)$ we have*: (i) $\langle\langle P'(d)f, f'\rangle\rangle = \langle\langle f, P'\cdot f'\rangle\rangle$. (ii) $\langle\langle f, P(d)f'\rangle\rangle = \langle\langle P\cdot f, f'\rangle\rangle$.

LEMMA 3.3. *The division maps* $P'^{-1}: P' \cdot \boldsymbol{P}_H(E) \subset \boldsymbol{F}(E) \to \boldsymbol{F}(E)$ *for* P' *in* $\boldsymbol{P}_H(E)$ *and* $P^{-1}: P \cdot \boldsymbol{P}_H(E') \subset \boldsymbol{F}_*(E') \to \boldsymbol{F}_*(E')$ *for* P *in* $\boldsymbol{P}_H(E')$ *are continuous.*

The continuity of the map P^{-1} is a consequence of the estimates $|||P_m|||_r\ |||Q|||_r \leqq |||P \cdot Q|||_r$, where P_m is the homogeneous term of highest order of P, Q any Hilbert-Schmidt polynomial on E_r', extending to infinite dimension an inequality of Trèves [**D1**, Lemma 3]. The continuity of P'^{-1} is proved by use of the analogous estimates on E_r, the inductive limit decomposition of $\boldsymbol{F}(E)$ and the verification that if a product of two polynomials on E is the restriction to E of a Hilbert-Schmidt polynomial on some E_r then the same holds for each factor.

PROOF OF THEOREM 3.2. Given P' in $\boldsymbol{P}_H(E)$ and f in $\boldsymbol{F}_*(E')$, a solution u in $\boldsymbol{F}_*(E)'$ of $P'(d)u = f$ is given by a Hahn-Banach extension $\langle\langle u, \ \rangle\rangle$ in $\boldsymbol{F}(E')$ of the continuous linear form $\langle\langle f, \ \rangle\rangle \circ P'^{-1}: P' \cdot \boldsymbol{P}_H(E) \to \boldsymbol{C}$, through Theorem 2.3 (iii) and Lemmas 3.2 and 3.3. This, with Theorem 3.1 (iii), proves part (i). Part (ii) is analogous.

4. **Examples.** (i) Let $\boldsymbol{S}_r(\boldsymbol{R})$, $r \geqq 0$, be the Hilbert subspaces of $L^2(\boldsymbol{R})$ of functions f for which $(f \mid h^r f) < \infty$ (L^2-scalar product), with $h = (x + d/dx)(x - d/dx)$, which have for projective limit the space $\boldsymbol{S}(\boldsymbol{R})$ of rapidly decreasing functions on $\boldsymbol{R}$, given in [**D2**] (except for a factor $\frac{1}{2}$): with $E_r = \boldsymbol{S}_r(\boldsymbol{R})$, the spaces $\boldsymbol{F}(\boldsymbol{S}(\boldsymbol{R}))$ and $\boldsymbol{F}_*(\boldsymbol{S}(\boldsymbol{R})') = \boldsymbol{F}(\boldsymbol{S}(\boldsymbol{R})')$ have properties similar to those of the tempered distributions and test functions in infinite dimension of [**KMP**]. The annihilation and creation operators are of the form $f \to \partial/\partial f$ (directional derivatives) and $f \mapsto \langle f, \ \rangle$, with the constant function 1 for vacuum element (cf. [**D3**]).

(ii) Let $\boldsymbol{F}_r(\boldsymbol{C})$ be the Hilbert spaces of entire functions isomorphic to the spaces $\boldsymbol{S}_r(\boldsymbol{R})$ given in [**Ba**] and [**D2**], $\boldsymbol{F}_\infty(\boldsymbol{C})$ and $\boldsymbol{F}_{-\infty}(\boldsymbol{C})$ their projective and inductive limits: with $E_r = \boldsymbol{F}_r(\boldsymbol{C})$ we obtain new representations $\boldsymbol{F}(\boldsymbol{F}_\infty(\boldsymbol{C}))$ and $\boldsymbol{F}_*(\boldsymbol{F}_{-\infty}(\boldsymbol{C})) = \boldsymbol{F}(\boldsymbol{F}_{-\infty}(\boldsymbol{C}))$ of the spaces studied in [**KMP**].

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