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AN APPLICATION OF MÖBIUS INVERSION TO A PROBLEM IN TOPOLOGICAL DYNAMICS

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1. **Introduction.** Let X be a topological space and let ϕ be a homeomorphism of X onto X. The pair (X, ϕ) is called a cascade. A nonempty subset M of X is a minimal subset of (X, ϕ) if M is closed, $\phi(M) = M$, and no proper subset of M has these properties. Equivalently M $(M \neq \varnothing)$ is minimal if and only if for every x in M we have $\mathrm{Cl}\{\phi^n(x):n \in Z\} = M$. A homomorphism of (X, ϕ) into (Y, ψ) is a continuous map θ of X into Y such that $\theta \circ \phi = \psi \circ \theta$.

Let $K = \{z \in C:|z| = 1\}$ and let (K, ϕ) be a cascade such that $\phi^n(x) = x$ implies $n = 0$. Then (K, ϕ) has exactly one minimal set which is either all of K or a Cantor subset C. We are only interested in the latter case, and we write

$$C = K \setminus \bigcup_{n=1}^{\infty} (a_n, b_n)$$

when (a_n, b_n) are counterclockwise open intervals in K and

$$[a_n, b_n] \cap [a_m, b_m] = \varnothing$$

whenever $n \neq m$. Note that $\phi[(a_i, b_i)] = (\phi(a_i), \phi(b_i)) = (a_j, b_j)$ for some $j \neq i$. Thus ϕ defines an equivalence relation on the complementary intervals $\{(a_n, b_n):n = 1, \ldots\}$. The restriction of ϕ to C is a homeomorphism of C onto C and produces a minimal cascade which we denote by (C, ϕ).

A cascade (X, ψ) on a compact Hausdorff space will be called an n-extension of (C, ϕ) if there exists an open n-to-one homomorphism of (X, ψ) onto (C, ϕ). If the number of equivalence classes of complementary intervals of C is finite, then for each positive integer n the number of isomorphism classes of minimal n-extensions is finite [**3**, Corollary 6.6]. Let $\mathscr{I}(C, \phi, n)$ denote this number. We consider the problem of determining $\mathscr{I}(C, \phi, n)$ or an asymptotic expression for it as n goes to infinity.

In the next section we present some combinatorial results which we applied to this problem, and in the last section we present our results on $\mathscr{I}(C, \phi, n)$. Proofs and tables of values can be found in [**1**].

2. **Combinatorial results.** Let N be a set with n elements, and let $\mathscr{S}_n$ be the symmetric group of all permutations acting on N. Let $\mathscr{S}_n^k$ be the set

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of all k-tuples of members of $\mathcal{S}_n$, and let T be the subset of $\mathcal{S}_n^k$ consisting of all $v = (v_1, \ldots, v_k)$ such that:

(1) $v_i \neq v_{i+1}$ and $v_k \neq v_1$,

(2) the components of v generate a transitive group.

Finally, let $\mathcal{S}_n$ act on T by conjugation. The number $M(n, k)$ is defined to be the number of orbits determined in T by this action.

Using Burnside's theorem [2] we obtain the following inequality:

$$\frac{\theta_k(n!)}{n!} - \sum_{j=1}^{[n/2]} \frac{\theta_k(i!\,(n-i)!)}{i!\,(n-i)!} \leqq M(n, k) \leqq \sum_{d \mid n} \frac{\theta_k[(n/d)^d d!]}{(n/d)^d d!}$$

where θ_k is the chromatic polynomial $\theta_k(s) = (s-1)^k + (-1)^k(s-1)$. From this we obtain

$$\lim_{k \to \infty} \frac{n!\, M(n, k)}{(n! - 1)^k} = 1, \qquad \lim_{n \to \infty} \frac{M(n, k)}{(n!)^{k-1}} = 1.$$

The cardinality $|T|$ of T is given by

$$n! \sum_{p(n)} \frac{(-1)^{\alpha_1 + \cdots + \alpha_m - 1}(\alpha_1 + \cdots + \alpha_m - 1)!}{1!^{\alpha_1} \cdots m!^{\alpha_m} \alpha_1! \cdots \alpha_m!} \theta_k(1!^{\alpha_1} \cdots m!^{\alpha_m})$$

where the sum is extended over all partitions of the integer n. This formula is derived by using Möbius inversion on the lattice of all partitions of the set N. The necessary Möbius function has been computed by Rota [4]. When n is prime we obtain the following exact formula for $M(n, k)$:

$$M(n, k) = |T| + \sum_{d \mid n} \mu(d)\theta_k(n/d),$$

where μ is the classical Möbius function.

Let T_0 be the subset of T such that $v_k = v_1^{-1} v_2 v_1$, let $\mathcal{S}_n$ act on T_0 by conjugation, and let $E(n, k)$ denote the number of orbits determined in T_0 by this action. The computation of $E(n, k)$ is similar to that for $M(n, k)$ except that a different chromatic polynomial is involved.

3. **Dynamical results.** The assumption that the number of equivalence classes of complementary intervals of C determined by the action of ϕ is finite will always be in force. This number will be denoted by m.

Let $\mathscr{C}(C, \phi, n)$ denote the number of minimal cohomology classes of (C, ϕ) for $\mathcal{S}_n$. Using the results in [3] one can pick canonical representatives for the minimal cohomology class of (C, ϕ) for $\mathcal{S}_n$ in such a way that they can be counted using the ideas in the previous section. First we obtain the inequality

$$M(n, m+1) \leqq \mathscr{C}(C, \phi, n) \leqq 1 + \sum_{k=2}^{m+1} \binom{m+1}{k} M(n, k)$$

and from this we obtain $\lim_{n\to\infty} \mathscr{C}(C, \phi, n)/(n!)^m = 1$.

THEOREM 1. *If the only automorphisms of* (C, ϕ) *are of the form* ϕ^k, *then* $\mathscr{I}(C, \phi, n) = \mathscr{C}(C, \phi, n)$ *and* $\lim_{n\to\infty} \mathscr{I}(C, \phi, n)/(n!)^m = 1$.

When $m = 1$ the minimal set (C, ϕ) is a Sturmian minimal set. In this case $\mathscr{I}(C, \phi, n) = 1 + M(n, 2)$ and when n is 2, 3, 4, 5, 6, 7 then $\mathscr{I}(C, \phi, n)$ is 3, 7, 26, 97, 624, 4157.

Finally, we obtain an exact formula.

THEOREM 2. *If all the automorphisms of* (C, ϕ) *are of the form* ϕ^k, *then*

$$\mathscr{I}(C, \phi, n) = 1 + \binom{m+1}{2} M(n, 2) + \sum_{k=3}^{m} \binom{m}{k} M(n, k) + \sum_{k=2}^{m} \binom{m}{k} [M(n, k+1) - E(n, k+1)].$$

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